# Lecture 1

## 1. Random matrices

Random matrix = matrix with random complex entries

**Ex**: (Wishart ensemble 1928) Define a $p \times n$ matrix $T$ with

$$T = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & \ddots & & \\ \vdots & & \ddots & \\ G_{p1} & & & G_{pn} \end{pmatrix}$$

with $G_{ij}$ independent, identically dist. (iid) and $G_{ij} \sim N_{\mathbb{R}}(0,1)$

and define

$$S = T T^{+} \leftarrow \text{a } p \times p \text{ random matrix.}$$

<u>Typical Q</u>: What is the typical dist. of the $p$ eigenvalues of $S$

<u>Other ex</u>: GUE, GOE, GSE, Ginibre, Wigner, etc.

## 3. A motivation for studying $U(N)$

### - Connections to $\zeta(s)$

$$U(N) = \{ g \in Mat_{N \times N}(\mathbb{C}) : g^* g = I \}$$

Recall spectral thm: If $A \in Mat_{N \times N}(\mathbb{C})$ is normal ($AA^* = A^* A$) then

$$A = \mathcal{U} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \mathcal{U}^*$$

for a unitary matrix $\mathcal{U}$, where $\lambda_1, \ldots, \lambda_n$ are e-values of $A$.

- Unitary $\Rightarrow$ normal, so all $g \in U(N)$ are diagonalizable
- $g^* g = I \Rightarrow \left(u \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_N \end{pmatrix} u^*\right)\left(u \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} u^*\right) = I$

$\Rightarrow |\lambda_i|^2 = 1$ for all $i$.

$\boxed{\text{so all } N \text{ eigenvalues of } U(N) \text{ lie on } \underbrace{\text{the unit circle}}_{S^1} \text{ of } \mathbb{C}.}$

Observation: Compared to $N$ points chosen uniformly and independently, on $S^1$, e-values of a random $g \in U(N)$ seem to distribute more rigidly (like picket fence).

Let $g \in U(N)$ have e-values $\{e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_N}\}$ with $\theta_j \in [-\frac{1}{2}, \frac{1}{2})$ for all $j$.

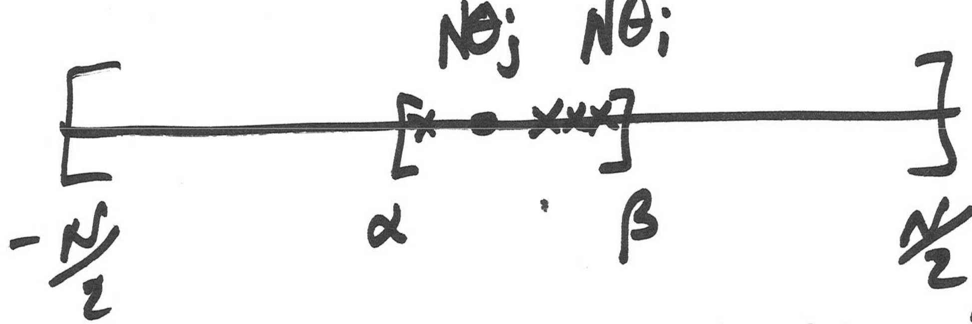
<u>Q</u>: How can we characterize the spacing between the $\theta_j$?

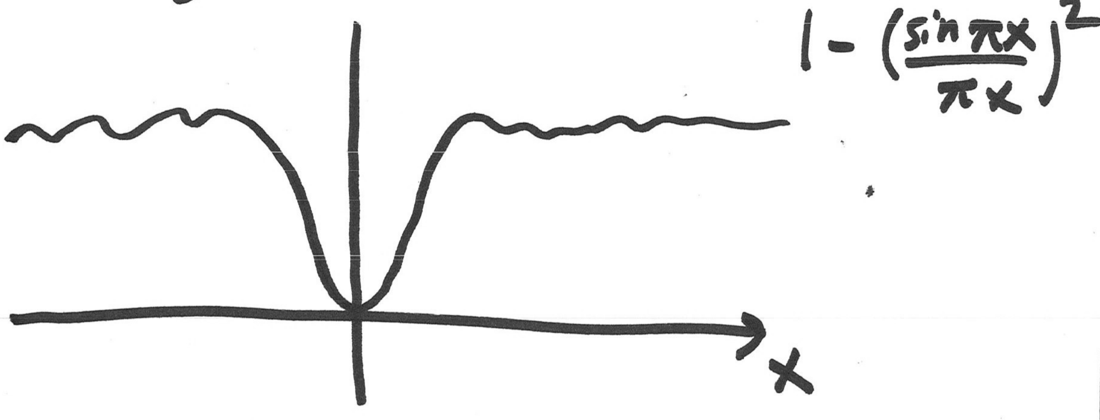
<u>Observation</u>: $N$ of the $\theta_j$ lie on an interval $[-\frac{1}{2}, \frac{1}{2})$. It's better to consider the points $\{N\theta_1, N\theta_2, \dots, N\theta_N\}$

(which have average spacing 1)

**Thm** (Dyson - Mehta) For any $\alpha < \beta$

$$\lim_{N \to \infty} \mathbb{E} \frac{1}{N} \# \left\{ (i,j) : i \neq j, \ N\theta_i - N\theta_j \in [\alpha, \beta] \right\} = \int_\alpha^\beta 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 dx$$



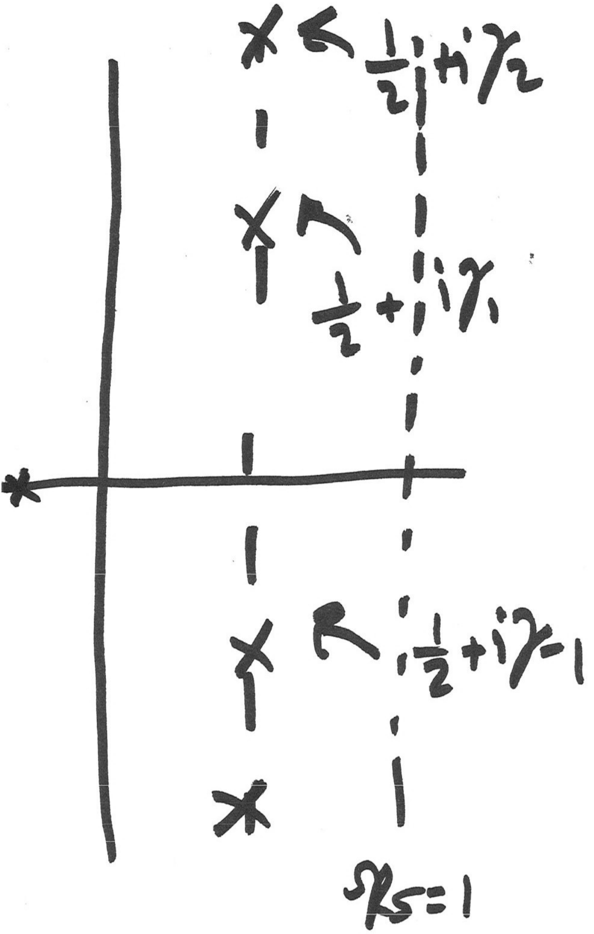




If the $\theta_i$ were independent this quantity would be

$$\int_\alpha^\beta 1 \, dx$$

## Zeta Zeros: $\zeta(s)$ a meromorphic function,

Zeros connected to primes




- Non-trivial zeros lie in $\Re s \in (0,1)$

- Riemann-Hypothesis: all non-trivial zeros are of form $\frac{1}{2}+i\gamma_n$.

- Density:
$$\#\{n : \gamma_n \in [T, 2T]\} \sim T \cdot \frac{\log T}{2\pi}$$

So for $\gamma_n \in [T, 2T]$, points $\left\{ \frac{\log T}{2\pi} \gamma_n \right\}$ have average spacing 1.

<u>Conj</u> (Montgomery):

$$\lim_{T \to \infty} \frac{1}{T \cdot \frac{\log T}{2\pi}} \# \left\{ (i,j) : i \neq j,\ \frac{\log T}{2\pi}(\gamma_i - \gamma_j) \in [\alpha, \beta],\ \gamma_i, \gamma_j \in [T, 2T] \right\}$$

$$= \int_\alpha^\beta 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 dx .$$