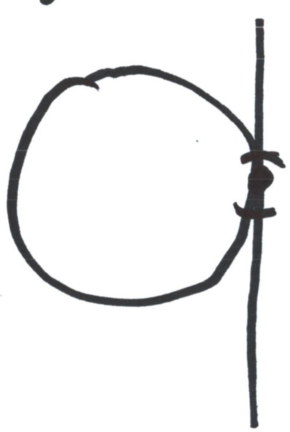
Lecture 10

0. Summary of last class

- e^X maps vector space $\mathfrak{g}$ (Lie algebra) to group G
- $e^X \approx I + X$ for small X, so $e^X e^Y \approx I + (X+Y)$ ← linearizes group action near identity
- e^X maps Lebesgue measure on $\mathfrak{g}$ to approx. of Haar measure on G,

  for small $X \in \mathfrak{g} \iff e^X$ near I.

$\mathcal{U}(1) \longrightarrow$



← $\mathfrak{u}(1)$

← Haar looks like Lebesgue

To be proved: Weyl formula for density of e-values ②

1. More preliminaries

Claim: $\mathcal{U}(N) \subseteq Mat_{N\times N}(\mathbb{C}) \cong \mathbb{R}^{2N^2}$ has dim. N^2 as a real manifold.

Why? Gram-Schmidt $\Rightarrow$ $\mathcal{U}(N)$ parametrized by $S^{2N-1} \times S^{2N-3} \times \cdots \times S^3 \times S^1$ of dim. N^2

Prop: If $k \in \mathcal{U}(N)$ is random with any distribution, and $m \in \mathcal{U}(N)$ is independent and Haar dist.

km has Haar dist; so does mk

Implies:

Claim: For $g \in \mathcal{U}(N)$ Haar dist.

$$g \overset{\text{dist.}}{\sim} h t h^{-1} \quad \text{with} \quad \begin{array}{l} h \in \mathcal{U}(N) \text{ Haar dist.} \\ t \text{ diagonal } N \times N, \end{array} \Big\rangle \text{ independent}$$

entries have same joint dist as e-values of g

Why? $g = k t k^{-1}$ for $k \in \mathcal{U}(N)$, dependent on t e-values

$m \in \mathcal{U}(N)$ Haar $\left\{ \begin{array}{l} \text{But conjugation } m g m^{-1} \text{ does not change dist.} \\ \text{of } g, \text{ and } m(k t k^{-1}) m^{-1} = (mk) t (mk)^{-1} \text{ has} \\ \text{dist } h t h^{-1} \text{ we want.} \end{array} \right.$

Claim: Dist. of e-values is given by a continuous prob. density function:

$$\mathbb{E} f(\omega_1, \dots, \omega_N) = \int_{[-\pi,\pi)^N} f(e^{i\theta_1}, \dots, e^{i\theta_N})\, \rho_N(\theta_1, \dots, \theta_N)\, d\theta_1 \cdots d\theta_N$$

e-values of random unitary

for some continuous ρ_N

Why? All combinations of e-values possible, use translation invariance of Haar.

Def: Diag = $N \times N$ diagonal matrices, real entries } real vector spaces dim. N (5)

Herm$'$ = $N \times N$ Hermitian matrices, vanishing entries on diagonal } dim. $N^2 - N$

<u>Lemma</u>: If $t = \mathrm{diag}(e^{it_1}, \ldots, e^{it_N})$ has all e^{it_j} distinct, for suff. small ε any $g \in \mathcal{U}(N)$ with $\|g - t\| \leq \varepsilon$ has a unique expression as

$$g = e^{iH} t e^{iD} e^{-iH} \quad \text{for} \quad \begin{array}{l} D \in \mathrm{Diag} \\ H \in \mathrm{Herm}' \end{array}$$

with $D, H = O(\varepsilon)$.

Computation: For $t = \mathrm{diag}(e^{it_1}, \ldots, e^{it_N})$

define

$$L : \mathrm{Herm}' \longrightarrow \mathrm{Herm}' \quad \text{by} \quad L(H) = tHt^{-1} - H$$

Then

$$|\det(L)| = \prod_{1 \le j < k \le N} |e^{it_j} - e^{it_k}|^2$$

(To be done soon: note if e^{it_j} distinct
$\Rightarrow \det(L) \neq 0$ and L invertible)

Proof of Lemma via computation:

For $H, D = O(\varepsilon)$,

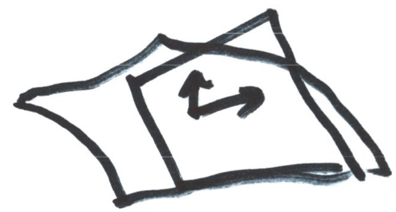
$$e^{iH} t e^{iD} e^{-iH} \approx (I+iH)t(I+iD)(I-iH) + O(\varepsilon^2)$$

$$\approx t + iHt + itD - itH + O(\varepsilon^2)$$

$$= t + it\left(D + t^{-1}Ht - H\right) + O(\varepsilon^2)$$

$$= t + it\left(\begin{pmatrix} I & \\ & L \end{pmatrix}\begin{pmatrix} D \\ H \end{pmatrix}\right) + O(\varepsilon^2)$$

$\begin{pmatrix} I & \\ & L \end{pmatrix}$ is invertible, so small changes in $\begin{pmatrix} D \\ H \end{pmatrix}$ take N^2 dim. space to N^2 dim. space



Inv. func. thm. $\Rightarrow$ g close to t has unique expression in D, H.

(What we've done is computed the $\underbrace{\text{Jacobian}}_{\det\begin{pmatrix} I & \\ & L \end{pmatrix}}$ of $\begin{pmatrix} D \\ H \end{pmatrix} \mapsto e^{iH} t e^{iD} e^{-iH}$ at $\begin{pmatrix} D \\ H \end{pmatrix} = 0$)