Lecture 11

1. A computation

Recall: Diag = $N \times N$ diagonal matrices, real entries $\cong \mathbb{R}^N$

Herm$'$ = $N \times N$ Hermitian matrices, vanishing entries in diagonal $\cong \mathbb{R}^{N^2-N}$

Computation claimed last class: For $t = \mathrm{diag}(e^{it_1}, \ldots, e^{it_N})$ define

$$L : \mathrm{Herm}' \longrightarrow \mathrm{Herm}', \quad H \mapsto tHt^{-1} - H$$

Then

$$|\det(L)| = \prod_{1 \le j < k \le N} |e^{it_j} - e^{it_k}|^2$$

<u>Proof</u>: Identify Herm$' \cong \mathbb{R}^{N^2-N}$

$\det(L) = \prod(\text{e-values})$, but have to place $\mathbb{R}^{N^2-N}$ in $\mathbb{C}^{N^2-N}$ to diagonalize

$$(\text{Herm})' + i(\text{Herm})' = \underset{\left(\frac{X+X^*}{2}\right)}{(\text{Herm})'} + \underset{\left(i\,\frac{X-X^*}{2i}\right)}{(\text{skew-Herm})'} = \underset{(X)}{\left\{\begin{matrix}\text{all complex}\\ \text{matrices with}\\ \text{vanishing diagonal}\end{matrix}\right\}}$$

a complex vector space dim N^2-N

$E_{jk} = 1$ in j^{th} column, k^{th} row, 0 otherwise; $j \neq k$.

$$L(E_{jk}) = t\,E_{jk}\,t^{-1} - E_{jk} = \left(e^{it_k}e^{-it_j} - 1\right)E_{jk}$$

multiplies row

multiplies columns

$$\det(L) = \prod_{j \neq k}\left(\frac{e^{it_k}}{e^{it_j}} - 1\right)$$

$$\Rightarrow |\det(L)| = \prod_{j<k} |e^{it_j} - e^{it_k}|^2$$

2. The Weyl integration formula

<u>Thm</u> (Weyl): For $g \in \mathcal{U}(N)$ Haar and $(\omega_1, \dots, \omega_N)$ the e-values (in uniformly random order)

$$\mathbb{E} f(\omega_1, \dots, \omega_N) = \int_{[-\pi,\pi)^N} f(e^{i\theta_1}, \dots, e^{i\theta_N})\, \rho_N(\theta_1, \dots, \theta_N)\, d\theta_1 \cdots d\theta_N$$

with $\rho_N(\theta_1, \dots, \theta_N) = \dfrac{1}{N!(2\pi)^N} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2$

for any continuous $f : (S^1)^N \to \mathbb{C}$.

<u>Proof</u>: Let $t^0 = \operatorname{diag}(e^{it_1^0}, \dots, e^{it_N^0})$, and $\varepsilon \to 0$.

Compute $\mathbb{P}(\|g - t^0\|_{HS} \leq \varepsilon)$ via two methods for $g \in \mathcal{U}(N)$.

Method 1:

$$\|g - t^0\| = \|\underbrace{(t^0)^{-1} g}_{\text{Haar dist., dist. does not depend on } t^0} - I\|$$

So

$$\mathbb{P}(\|g - t_0\| \leq \varepsilon) = \mathbb{P}(\|g - I\| \leq \varepsilon)$$

$$\sim C_1 \varepsilon^{N^2}$$

for a constant c_1 depending on N, but not t^0.

Method 2 : $\|g - t^0\| \le \varepsilon$

$$\Rightarrow g = h t h^{-1} \text{ for } \underbrace{t = t^0 e^{iD},\ D \in \text{Diag},\ D = O(\varepsilon) \quad h = e^{iH},\ H \in \text{Herm}';\ H = O(\varepsilon)}_{\text{Uniquely determined}}$$

Claim I: As $\varepsilon \to 0$, if $\mathcal{U} \subset \text{Diag}$, $\mathcal{U}$ is inside ε-ball of 0

$$\mathbb{P}(D \in \mathcal{U}) = (c_2 + o(1))\, \rho_N(t^0) \int_{\mathcal{U}} dx_1 \cdots dx_N$$

Lesbesgue meas. weighted by e-value density

Why? t has the same dist as e-values of g, so for $t \approx t^0$, dist. of D must be asymptotically proportional to $\rho_N(t^0) \cdot$ (Lesbesgue measure).

Claim II: As $\varepsilon \to 0$, if $V \subset \mathrm{Herm}'$,

V inside ε-ball of 0,

$$\mathbb{P}(H \in V) = (C_3 + o(1)) \int_V dy_1 \cdots dy_{N^2-N}$$

Lebesgue meas., no dependence on t^0.

Moreover, dist of D and H are independent.

(To be justified soon.)

Then for $\|g - t^0\| \leq \varepsilon$, (7)

$$g - t^0 = e^{iH} t^0 e^{iD} e^{-iH} - t^0$$

$$= i t^0 \left(D + (t^0)^{-1} H (t^0) - H\right) + O(\varepsilon^2)$$

← calc. last class.

so

$$\mathbb{P}(\|g - t^0\| \leq \varepsilon) = \mathbb{P}\left(\|D + (t_0)^{-1} H (t^0) - H\| \leq \varepsilon + O(\varepsilon^2)\right)$$

$$\sim C_4 \left(\rho_N(t^0) \cdot \varepsilon^N\right) \left(\frac{1}{|\det(L)|} \varepsilon^{N^2 - N}\right)$$

claim I, N dim Lebesgue, weighted by density

claim II, $N^2 - N$ dim Lebesgue, passed through L

$$= C_4 \frac{\rho_N(t^0)}{|\det(L)|} \cdot \varepsilon^{N^2}$$

Methods 1+2 $\Rightarrow$ $\rho_N(t^0) = (\text{constant}) \cdot |\det(L)|$

depends on N

$\Rightarrow$ Weyl, with $\dfrac{1}{N!\,(2\pi)^N}$ not yet determined

To be determined in HW by $\int \rho_N = 1$ $\square$

Why claim II? Note if g has e-values t, conjugating matrix determined up to right mult. by diagonal matrix:

$$g = (e^{iH} r)\, t\, (e^{iH} r)^{-1}$$ with r diagonal, entries freely chosen on unit circle

Hence $t \overset{dist}{\sim}$ e-values, $e^{iH} r \overset{dist.}{\sim}$ Haar on $U(N)$, independent

Another Jacobian comp. reveals $e^{iH} r \overset{dist.}{\sim}$ Haar $\Leftarrow\Rightarrow$ r has uniform entries on $(S^1)^N$, H locally looks Lebesgue, independent