Lecture 12

1. Constraints on e-values

The unitary group

$$\text{Possible sets of e-values for } g \in \mathcal{U}(N) = \left\{ \left(e^{i\theta_1}, \ldots, e^{i\theta_N}\right) : \theta_1, \ldots, \theta_N \in [0, 2\pi) \right\}$$

$$\text{Possible sets of e-values for } g \in SU(N) = \left\{ \left(e^{i\theta_1}, \ldots, e^{i\theta_N}\right) : e^{i\theta_1} \cdots e^{i\theta_N} = 1 \right\}$$

The orthogonal group

Possible sets of e-values for $g \in SO(2N)$ $= \left\{ \left(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N} \right) : \theta_1, \dots, \theta_N \in [0, \pi] \right\}$

Possible sets of e-values for $g \in SO(2N+1)$ $= \left\{ \left(1, e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N} \right) : \theta_1, \dots, \theta_N \in [0, \pi] \right\}$

Possible sets of e-values for $g \in O(N)$ $= \left\{ \left(\underbrace{\pm 1, \pm 1, \dots, \pm 1}_{\text{same parity as } N}, e^{i\theta_1}, \dots, e^{i\theta_k}, e^{-i\theta_1}, \dots, e^{-i\theta_k} \right) : \theta_i \in (0, \pi) \right\}$

<u>Why?</u> $SO(M) \subset O(M) \subset U(M)$

$\Rightarrow M$ e-values, all on unit circle

$O(M)$ { If $g \in O(M)$, $\det(\lambda I - g) = P(\lambda)$ ← polynomial, with real coeff.

each e-value λ must be real (thus ± 1), or have a conjugate pair.

$SO(2N)$ { If $M = 2N$, and $\det(g) = 1$, e-values $= -1$ must occur in pairs

so e-values $= 1$ occur in ~~pairs~~ pairs too

$SO(2N+1)$ { If $M = 2N+1$, and $\det(g) = 1$, e-values $= -1$ occur in pairs, leaves one extra e-value $= 1$ occuring alone.

(Show all such e-value sets realized via block rotation matrices)

The symplectic group

Possible sets of e-values for $g \in Sp(2N)$ $= \left\{ \left(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}\right) : \theta_1, \dots, \theta_N \in [0, \pi] \right\}$

Why? $Sp(2N) \subset \mathcal{U}(2N)$, so $2N$ e-values on unit circle

Recall $g \in Sp(2N) \iff \{ g \in \mathcal{U}(2N)$ and $Jg = \bar{g} J \}$ for $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

Note if $g\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$ $\leftarrow \in \mathbb{C}^N$ then

$$g\begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = J^{-1} \bar{g} J \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = J^{-1} \bar{g} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \bar{\lambda} J^{-1} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \bar{\lambda} \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix}$$

So e-value $\lambda \Rightarrow$ e-value $\bar{\lambda}$ and evalues occur in conjugate pairs.

This pattern realized via
$$\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_N},e^{-i\theta_1},\dots,e^{-i\theta_N})$$

<u>Cor</u> (claimed earlier): If $g \in Sp(2N)$, then $\det(g)=1$.

2. Distribution of e-values

<u>Def</u>: $f: G \to \mathbb{C}$ is a class function if it is constant on conjugacy classes: $f(h^{-1}gh)=f(g)$ $\forall g, h$.

For $G \subset \mathcal{U}(N)$, conjugacy classes determined by e-values, so:

class function $\Longleftrightarrow$ function of set of e-values

<u>Thm</u> (Weyl for $\mathcal{U}(N)$): For $g \in \mathcal{U}(N)$, Haar dist., and f a class function

$$\mathbb{E} f(g) = \frac{1}{N!} \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} f\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_N} \end{pmatrix} \cdot \prod_{1 \le j < k \le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 d\theta_1 \cdots d\theta_N$$

<u>Note</u>: also write

$$\mathbb{E} f(g) = \int_{U(N)} f(g)\, dg, \quad \text{for } g \in \mathcal{U}(N) \text{ Haar dist.}$$