Lecture 13

1. Distribution of e-values (cont.)

Thm (Weyl for $SO(2N)$, $SO(2N+1)$) For f a class function

$$\int_{SO(2N)} f(g)\,dg = \frac{1}{N!\,2^{N-1}}\frac{1}{\pi^N}\int_{[0,\pi]^N} f\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_N} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_N}\end{pmatrix} V_N(\theta_1,\dots,\theta_N)\,d\theta_1\cdots d\theta_N$$

for $V_N(\theta_1,\dots,\theta_N) = \prod_{1\le j<k\le N}\left(2\cos\theta_j - 2\cos\theta_k\right)^2$

$$\int_{SO(2N+1)} f(g)\,dg = \frac{2^N}{N!}\frac{1}{\pi^N}\int_{[0,\pi]^N} f\begin{pmatrix} 1 & & & & & & \\ & e^{i\theta_1} & & & & & \\ & & \ddots & & & & \\ & & & e^{i\theta_N} & & & \\ & & & & e^{-i\theta_1} & & \\ & & & & & \ddots & \\ & & & & & & e^{-i\theta_N}\end{pmatrix} V_N(\theta_1,\dots,\theta_N)\prod_{j=1}^{N}\sin^2\!\left(\frac{\theta_j}{2}\right)d^N\theta$$

Thm (Weyl for $Sp(2N)$)

$$\int_{Sp(2N)} f(g)\,dg = \frac{2^N}{N!}\frac{1}{\pi^N}\int_{[0,\pi]^N} f\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_N} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_N}\end{pmatrix} V_N(\theta_1, \dots, \theta_N) \cdot \prod_{j=1}^N \sin^2(\theta_j)\, d\theta_1 \cdots d\theta_N$$

(Proved via computation of determinants / Jacobians, as for $U(N)$.)

2. Determinantal formulas

Recall Vandermonde identity

$$\prod_{1\le j<k\le N} (z_j - z_k) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & & & \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1}\end{pmatrix}$$

For $\mathcal{U}(N)$

Thus if $K_N(x,y) = \sum_{j=0}^{N-1} e^{ijx} e^{-ijy}$

$$\prod_{j<k} \left|e^{i\theta_j} - e^{i\theta_k}\right|^2 = \det \begin{pmatrix} 1 & e^{i\theta_1} & \cdots & e^{i(N-1)\theta_1} \\ 1 & \vdots & & \vdots \\ \vdots & \vdots & & \\ 1 & e^{i\theta_N} & & e^{i(N-1)\theta_N} \end{pmatrix}$$

(Useful in HW)

$$\cdot \det \begin{pmatrix} 1 & \cdots & 1 \\ e^{-i\theta_1} & \cdots & e^{-i\theta_N} \\ \vdots & & \\ e^{-i(N-1)\theta_1} & \cdots & e^{-i(N-1)\theta_N} \end{pmatrix}$$

$$= \det \begin{pmatrix} K_N(\theta_1,\theta_1) & K_N(\theta_1,\theta_2) & \cdots \\ K_N(\theta_2,\theta_1) & \ddots & \\ \vdots & & \\ & & K_N(\theta_N,\theta_N) \end{pmatrix}$$

Observation:

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_N \\ \vdots & & \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix} = \det\begin{pmatrix} 1 & \cdots & 1 \\ P_1(z_1) & \cdots & P_1(z_N) \\ \vdots & & \\ P_{N-1}(z_1) & \cdots & P_{N-1}(z_N) \end{pmatrix}$$

for <u>any</u> collection of monic polynomials $1, P_1, \ldots, P_{N-1}$ with $\deg P_j = j$. (By multilinearity of det.)

for $Sp(2N)$

$$\prod_{j<k}(2\cos\theta_j - 2\cos\theta_k)\prod_{k=1}^{N}\sin(\theta_k)$$

$$= \det_{N\times N}\left(P_{j-1}(2\cos\theta_k)\right)\prod_{k=1}^{N}\sin(\theta_k)$$

$$= \det_{N\times N}\left(P_{j-1}(2\cos\theta_k)\sin\theta_k\right)$$

for any monic poly P_j

Fact: $\sin(j\theta) = P_{j-1}(2\cos\theta)\sin\theta$ for some monic P_{j-1} of deg. $j-1$

(In fact: $P_{j-1}(z) = U_{j-1}(z/2)$ ← Chebyshev poly. of 2^{nd} kind)

Thus if $L_N(x,y) = \sum_{j=1}^{N} 2\sin(jx)\sin(jy)$

$$2^N \prod_{j<k}(2\cos\theta_j - 2\cos\theta_k)^2 \prod_{k=1}^{N} \sin^2\theta_k = 2^N \det_{N\times N}(\sin(j\theta_k)) \cdot \det_{N\times N}(\sin(j\theta_k))$$

same trick as for $U(N)$, multiply each row by 2 at end $\longrightarrow$ $= \det_{N\times N}(L_N(\theta_j,\theta_k))$.

[So e-values of $\mathcal{U}(N)$ controlled by a random choice of

$$(\theta_1, .., \theta_N) \in [-\pi, \pi]^N$$

with density function

$$\frac{1}{N!} \det_{N\times N}\left(K_N(\theta_j, \theta_k)\right) \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}$$

[e-values of $Sp(2N)$ controlled by

$$(\theta_1, .., \theta_N) \in [0, \pi]^N$$

with density

$$\frac{1}{N!} \det_{N\times N}\left(L_N(\theta_j, \theta_k)\right) \frac{d\theta_1}{\pi} \cdots \frac{d\theta_N}{\pi}$$

Similar formula for $SO(2N)$, $SO(2N+1)$ $\longleftarrow$ Table p.72 of Meckes

3. Point process = random configurations of points in space

Recall: to describe real-valued random variable X,
We take the space $\mathbb{R}$,
and form the σ-algebra $\mathcal{B}$ of Borel sets,
with a basis of open sets,

and for all $A \in \mathcal{B}$, must have a consistent*
collection of answers to $\mathbb{P}(X \in A)$

(* means $\mu(A) = \mathbb{P}(X \in A)$ is a measure
with $\mu(\mathbb{R}) = 1$)

To describe a point process ξ on Λ (for $\Lambda = \mathbb{R}$ or an interval)

we take the space of locally finite configurations

$$\mathcal{C} = \{ \{x_i\} : \#\{x_i \in K\} = \#_K(x) < \infty \text{ for all compact } K \}$$

finite or infinite sets in $\mathbb{R}$

and form a σ-algebra $\mathcal{F}$ with a basis consisting of 'cylinder sets'

$$C_n^B := \{ x \in \mathcal{C} : \#_B(x) = n \} \quad n = 0, 1, \ldots \text{ for } B \subseteq \Lambda \text{ a Borel set}$$

A point process ξ is a consistent* collection of answers to $\mathbb{P}(\xi \in F)$ for all $F \in \mathcal{F}$.

(*i.e. a prob. measure on $(\mathcal{C}, \mathcal{F})$)

e.g. Want to be able to answer

$$\mathbb{P}\left(\#_{(\frac{\pi}{2}, e)}(x) = 5, \#_{(\pi, 2\pi)}(x) = 7, \#_{(10, 10.5)}(x) = 0 \right)$$