Lecture 14

1. Foundations of point processes

Let $\Lambda = \mathbb{R}$, or an interval $[a,b]$

$\mathcal{C}$ = "space of locally finite configurations"

$:= \{ \{x_i\} : \#_K(x) = \#\{x_i \in K\} < \infty$ for all compact $K\}$

$\{x_i\}$ ← finite or countable collection of elements of Λ

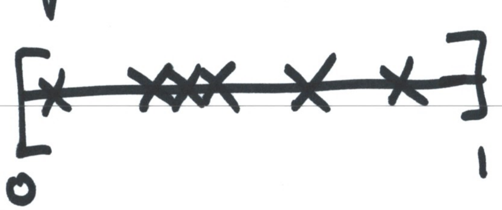
$C_n^B := \{ x \in \mathcal{C} : \#_B(x) = n \}$, $\mathcal{F}$ the σ-algebra on $\mathcal{C}$ generated by C_n^B, for $n = 0, 1, 2, \ldots$, B a Borel subset of Λ

Def: A <u>point process</u> ξ is a random element of $(\mathcal{C}, \mathcal{F})$.

Note: Any point process ξ induces a probability measure (the "distribution of ξ") on $(\mathcal{C}, \mathcal{F})$ via $\mathbb{P}(\xi \in F)$ for $F \in \mathcal{F}$, and conversely any prob. measure on $(\mathcal{C}, \mathcal{F})$ defines a random element. (Analogy to random variables on $\mathbb{R}$.)

Ex: Poisson process on $[0,1]$ of intensity λ.

Intuition: calls into an hour long telethon, with call at time t a point at $t \in [0,1]$, with callers independent, and $\lambda = \#$ calls expected.



One construction: Calls occur in succession, after each call, time τ to next call is independent r.v. with $\mathbb{P}(\tau > s) = e^{-\lambda s}$.

Another construction: Point process ξ with

$$\mathbb{P}\left(C_{n_1}^{B_1} \cap C_{n_2}^{B_2} \cap \cdots \cap C_{n_k}^{B_k}\right) = \prod_{j=1}^{k} \frac{(\lambda |B_i|)^{n_i}}{n_i!} e^{-\lambda |B_i|}$$

Poisson densities

for $B_1, \ldots, B_k \subset [0,1]$ disjoint.

Does this define a prob. measure on $(\mathcal{C}, \mathcal{F})$? ④

First test: additivity example – if $A = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$

$$\mathbb{P}(\#_A = n) \overset{?}{=} \mathbb{P}(\#_{B_1} = n \cap \#_{B_2} = 0) + \mathbb{P}(\#_{B_1} = n-1 \cap \#_{B_2} = 1) + \cdots + \mathbb{P}(\#_{B_1} = 0 \cap \#_{B_2} = n)$$

This is

$$\frac{1}{n!}\left(\lambda(|B_1| + |B_2|)\right)^n = \sum_{j=0}^{n} \frac{1}{(n-j)!}(\lambda|B_1|)^{n-j} e^{-\lambda|B_1|} \frac{1}{j!}(\lambda|B_2|)^j e^{-\lambda|B_2|}$$

True by binomial thm!

<u>Fact 1 from probability</u>: Given quantities

$\mathbb{P}(C_{n_1}^{B_1} \cap \cdots \cap C_{n_k}^{B_k}) \in [0,1]$ (for all $k \geq 1$)

that satisfy finite additivity and

$$\sum_{n_k=0}^{\infty} \mathbb{P}(C_{n_1}^{B_1} \cap \cdots \cap C_{n_k}^{B_k}) = \mathbb{P}(C_{n_1}^{B_1} \cap \cdots \cap C_{n_{k-1}}^{B_{k-1}})$$

there exists a unique point process ξ

with these probabilities.

(Follows via Kolmogorov's consistency thm.)

Fact 2 from probability: If ξ and ξ' are two point processes and if for all $k \geq 1$ and all collections of open intervals $I_1, I_2, \ldots, I_k \subset \Lambda$ the joint distributions

$$\left(\#_{I_1}(\xi), \#_{I_2}(\xi), \ldots, \#_{I_k}(\xi)\right) \text{ and } \left(\#_{I_1}(\xi'), \ldots, \#_{I_k}(\xi')\right)$$

agree, then ξ and ξ' have the same dist.

(Follows from above fact 1, or more simply)

Fact 3 from probability : For ξ and ξ',

let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous and compactly supported function, and

$$\sum_i \eta(\xi_i) \quad \text{and} \quad \sum_i \eta(\xi_i')$$

If for all such η these rv have the same dist., then ξ and ξ' have the same dist.

<u>Idea of proof</u>: Can pass from such

$\eta(x)$ to $\quad f(x) = \sum_{j=1}^{k} \alpha_i \mathbb{1}_{I_j}(x)$

Fact 3 $\iff$ For all scalars α_i $\quad \sum_i \sum_{j=1}^{k} \alpha_j \mathbb{1}_{I_j}(\xi_i)$

$$= \sum_{j=1}^{k} \alpha_j \#_{I_j}(\xi)$$

has same dist as for ξ'

$\overset{\text{Cramer-Wold}}{\iff}$ $\underbrace{\left(\#_{I_1}(\xi), \ldots, \#_{I_k}(\xi)\right) \text{ has same joint dist as } \left(\#_{I_1}(\xi'), \ldots, \#_{I_k}(\xi')\right)}_{\text{Fact 2}}$