Lecture 15

1. First intensity measure for point processes

Recall: point process ξ = random point configurations

questions in the σ-algebra: what is prob. $\underbrace{\text{n points lie in some set B}}_{C_n^B = \{\xi : \#_B(\xi) = n\}}$

Recall:

Poisson with intensity λ on $[0,1]$: $\mathbb{P}(C_n^B) = \frac{(\lambda|B|)^n}{n!} e^{-\lambda|B|}$, events $C_{n_1}^{B_1}, \dots, C_{n_k}^{B_k}$ independent if $B_1, \dots, B_k$ disjoint

<u>Def</u>: For ξ a point process with configurations in $\Lambda = \mathbb{R}$ or $[\alpha, \beta]$ if

$$\mathbb{E} \sum_i \eta(\xi_i) = \int_\Lambda \eta(t)\, d\mu_1(t) \quad \forall \eta \in C_c(\Lambda)$$

then μ_1 is the *first intensity measure* (or 1-level density) of ξ.

(Note: as long as $\mathbb{E} \sum \eta(\xi_i) < \infty$ for all $\eta \in C_c(\Lambda)$, $\eta \geq 0$, then μ_1 always exists by Riesz representation thm.)

<u>Ex</u>: For Poisson process in $[0,1]$ of intensity λ, suppose $B \subset [0,1]$

$$\mathbb{E} \sum_i \mathbb{1}_B(\xi_i) = \mathbb{E}\, \#_B(\xi) = \sum_{n=0}^{\infty} n \cdot \mathbb{P}(\#_B = n)$$

$$= \sum_{n=0}^{\infty} n \cdot \frac{(\lambda |B|)^n}{n!} e^{-\lambda |B|} = \lambda |B|$$

$$= \lambda \int \mathbb{1}_B \, dt$$

$$\Rightarrow \mathbb{E} \sum \eta(\xi_i) = \lambda \int_0^1 \eta(t)\, dt$$ (by approx. η above and below by step functions)

So $\boxed{d\mu_1(t) = \lambda\, dt}$ (prob. of a point being in $[t, t+dt]$ is $\lambda\, dt$)

<u>Ex</u>: For $\{\vartheta_1, \dots, \vartheta_N\}$ are the e-angles of $g \in \mathcal{U}(N)$, $\vartheta_i \in [-\frac{1}{2}, \frac{1}{2})$ (so $e^{i2\pi\vartheta_j}$ are e-values)

$$\mathbb{E} \sum_i \eta(\vartheta_i) = N \int_{-1/2}^{1/2} \eta(t)\,dt$$

so $d\mu_1(t) = N\,dt$.

<u>Why</u>? e-angles of $\mathcal{U}(N)$ are <u>translation-invariant</u>

$$e^{i\tau} g \overset{\text{dis.}}{\sim} g \implies \{\vartheta_1 + \tau, \dots, \vartheta_N + \tau\} \overset{\text{dis.}}{\sim} \{\vartheta_1, \dots, \vartheta_N\}$$

$$\implies d\mu_1(t+\tau) = d\mu_1(t)$$

(fixed τ)

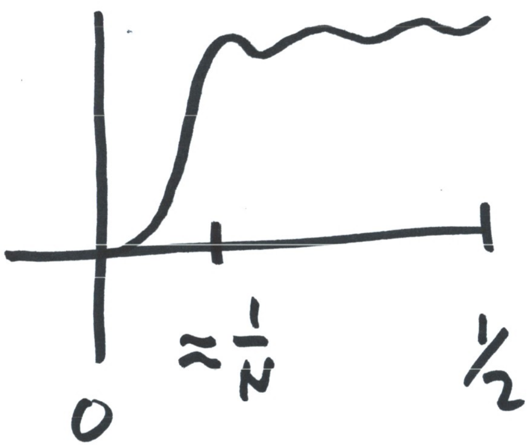
$$\implies d\mu_1(t) = c \cdot dt$$

For $\eta(x) = 1$, $\mathbb{E} \sum \eta(\vartheta_i) = N \implies c = N$.

<u>Ex</u>: For first N e-angles $\{\vartheta_1, \dots, \vartheta_N\}$ of $g \in Sp(2N)$ with $\vartheta_i \in [0, \tfrac{1}{2}]$ (so e-values are $e^{\pm i 2\pi \vartheta_j}$)

$$d\mu_1(t) = 2N\left(1 + \frac{1}{2N} - \frac{\sin\left((2N+1)\,2\pi t\right)}{2N \sin(2\pi t)}\right) dt$$

More work needed!



To come!

2. Joint intensities

Motivation: higher moments $\mathbb{E}\left(\sum_i \eta(\xi_i)\right)^k$

Def: The *k^{th} joint intensity measure* (or *k-point correlation measure*) *k-level density* or is $d\mu_k$ if

$$\mathbb{E}\sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}} \eta(\xi_{i_1},\ldots,\xi_{i_k}) = \int_{\Lambda^k} \eta(t_1,..,t_k)\, d\mu_k(t_1,..,t_k)$$

Often $d\mu_k(t_1,..,t_k) = \rho_k(t_1,..,t_k)\,dt_1 \cdots d_k$

Then ρ_k is the *k^{th} joint intensity function* (or *correlation function*)

Think of joint intensities as being analogous to moments of a rv.

Even closer to 'factorial moments'

Note:

$$\mathbb{E}\Big(\sum_i \eta(\xi_i)\Big)^2 = \mathbb{E}\sum_{i,j} \eta(\xi_i)\eta(\xi_j)$$

$$= \mathbb{E}\sum_{\substack{i,j \\ \text{distinct}}} \eta(\xi_i)\eta(\xi_j) + \mathbb{E}\sum_i \eta(\xi_i)^2$$

So joint intensities encode moments of linear statistics and vice-versa.

(Similar formulas for higher moments)

<u>Ex</u>: Poisson process in $[0,1]$ of intensity λ, $B \subset [0,1]$

$$\mathbb{E}(\#_B)^2 = \mathbb{E}\sum_{\substack{i,j \\ \text{distinct}}} \mathbb{1}_B(\xi_i)\mathbb{1}_B(\xi_j) + \underbrace{\mathbb{E}\sum_i \mathbb{1}_B(\xi_i)^2}_{\mathbb{E}\#_B}$$

$$\Rightarrow \int_{B^2} d\mu_2(t_1,t_2) = \mathbb{E}\#_B(\#_B - 1) = \lambda^2 |B|^2$$

a formula for $\#_B$ a Poisson rv

A similar analysis $\Rightarrow \int_{B_1 \times B_2} d\mu_2(t_1,t_2) = \lambda^2 |B_1|\cdot|B_2|$

$$\Rightarrow d\mu_2 = \lambda^2\, dt_1\, dt_2$$

In fact :

$$\mathbb{E}\, \#_B(\#_B - 1) \cdots (\#_B - (k-1)) = \lambda^k |B|^k$$

and

$$d\mu_k = \lambda^k \, dt_1 \cdots dt_k$$