Lecture 16

1. Existence and uniqueness of point processes

Recall

$$\mathbb{E}\sum_{\substack{j_1,\dots,j_k\\ \text{distinct}}} \eta(\xi_{j_1},\dots,\xi_{j_k}) = \int_{\Lambda^k} \eta \, d\mu_k$$

the k-th joint intensity

$\Lambda = \mathbb{R}$ or $[\alpha,\beta]$

Usually $d\mu_k(x) = \rho_k(x_1,\dots,x_k)\,dx_1\cdots dx_k$

point process $\longleftrightarrow$ joint intensity

analogous to

random variable $\longleftrightarrow$ moments

<u>Q1</u> (Uniqueness) If ξ and ξ' have the same joint intensities, do they have the same distribution?

<u>A1</u>: If $\mathbb{P}(\#_E(\xi) > t) = O(e^{-ct})$ for some constant c, and all compact intervals E, then yes. (See HKPV, Remark 1.24)

<u>A quick observation</u>: μ_k must be a positive measure on Λ^k, $d\mu_k(x_1, .., x_k)$ must be symmetric in $x_1, .., x_k$.

Q2 (Existence): Given a sequence of symmetric, positive 'joint intensities', does there exist a point process corresponding to them?

A2: No! Need a more general positivity criterion, and it's not very effective to check... (cf. the Hamburger moment problem for random var.)

2. Point processes with always N points

Prop: If a point process on Λ ($=\mathbb{R}$ or $=[\alpha,\beta]$) consists of always N points, then

$$\int_\Lambda \rho_{k+1}(x_1,\ldots,x_k,x_{k+1})\,dx_k = (N-k)\cdot\rho_k(x_1,\ldots,x_k)$$

for $k=1,2,\ldots,N-1$.

Remark 1: There is a version for general $d\mu_k$, not just $\rho_k\,dx_1\ldots dx_k$.

Remark 2: In general can't go from ρ_{k+1} to ρ_k
– special to having always N points!

<u>Proof</u>: Let $\eta(x_1, \ldots, x_k, x_{k+1}) = h(x_1, \ldots, x_k)$.

Then

$$\mathbb{E} \sum_{\substack{\xi_{j_1}, \ldots, \xi_{j_{k+1}} \\ \text{distinct}}} \eta(\xi_{j_1}, \ldots, \xi_{j_{k+1}}) = \mathbb{E}(N-k) \sum_{\substack{\xi_{j_1}, \ldots, \xi_{j_k} \\ \text{distinct}}} h(\xi_{j_1}, \ldots, \xi_{j_k})$$

$$= \int_{\Lambda^k} h(x) \left[(N-k) \rho_k(x) \right] d^k x$$

But this is also

$$= \int_{\Lambda^{k+1}} \eta(x_1, \ldots, x_{k+1}) \rho_{k+1}(x_1, \ldots, x_{k+1}) \, d^{k+1} x$$

$$= \int_{\Lambda^k} \int_{\Lambda} h(x_1, \ldots, x_k) \rho_{k+1}(x_1, \ldots, x_k, x_{k+1}) \, dx_{k+1} \, d^k x$$

$$= \int_{\Lambda^k} h(x) \left(\int_{\Lambda} \rho_{k+1}(x_1, \ldots, x_k, x_{k+1})\, dx_{k+1} \right) d^k x$$

As h is arbitrary, this proves it. $\square$

<u>Prop</u>: For a point process with always N points,

$$\frac{1}{N!} \rho_N(x_1, \ldots, x_N)$$

is a prob. density on Λ^N.

(Proof is an ex. for you.)

3. Determinantal point processes

A point process on Λ is determinantal if

$$\rho_k(x_1, \ldots, x_k) = \det_{k \times k}\left(K(x_i, x_j)\right)$$

for all $k \geq 1$

for some function K. (meeting a few conditions to come later.)

Def: Point process is determinantal projection (of rank N) if it is determinantal as above and

$$K(x,y) = \sum_{j=1}^{N} \varphi_j(x)\overline{\varphi_j(y)} \qquad (*)$$

for $\{\varphi_1, \ldots, \varphi_N\}$ orthonormal in $L^2(\Lambda)$.

<u>Ex</u>: e-angles of $\mathcal{U}(N)$, $Sp(2N)$.

Check this yourself from formulas earlier!

for N-point correlations, (all correlations to come ...)

Lemma (Gaudin): For K as in $(*)$

$$\int_{\Lambda} \det_{(k+1)\times(k+1)} \left(K(x_i, x_j)\right) dx_{k+1} = (N-k) \det_{k\times k} \left(K(x_i, x_j)\right)$$

(Proof next time.)

<u>Cor</u>: For K as in $(*)$, if we have N points $(x_1, \ldots, x_N) \in \Lambda^N$ with joint pdf

$$\frac{1}{N!} \rho_N(x_1, \ldots, x_N)$$

with

$$\rho_N(x_1, \ldots, x_N) = \det_{N \times N}\Big(K(x_i, x_j)\Big),$$

then these points $\{x_1, \ldots, x_N\}$ comprise a determinantal projection process with N points.

<u>Proof</u> : Want to verify

$$\rho_k(x_1, \dots, x_k) = \det_{k \times k} \left(K(x_i, x_j) \right) \quad \text{for } k = 1, \dots, N$$

k-point correlation functions

earlier prop.

- True for $k = N$
- For $k = N-1$

$$(N - (N-1))\,\rho_{N-1}(x_1, \dots, x_{N-1}) = \int_\Lambda \rho_N(x_1, \dots, x_N)\, dx_N$$

But then Gaudin's Lemma $\Rightarrow$ $\rho_{N-1}(x) = \det_{(N-1)\times(N-1)} \left(K(x_i, x_j) \right)$

- Continue down inductively for $k = N-2, \dots, 1$. $\square$