Lecture 17

1. Determinantal projection point processes

Suppose

$$(*) \qquad K(x,y) = \sum_{j=1}^{N} \varphi_j(x)\overline{\varphi_j(y)} \quad \text{for } \{\varphi_1, \dots, \varphi_N\} \text{ orthonormal in } L^2(\Lambda)$$

$\Lambda = \mathbb{R}$ or $[\alpha, \beta]$, $L^2(\Lambda) = L^2(\Lambda, dx)$

K defines an operator $\underline{K} : L^2(\Lambda) \to L^2(\Lambda)$ by

$$(\underline{K} f)(x) = \int_\Lambda K(x,y)\, f(y)\, dy$$

<u>Prop</u>: If $\mathcal{H} = \operatorname{span}\{\varphi_1, \ldots, \varphi_N\} \subseteq L^2(\Lambda)$,

then

a) $\underline{K} f = \operatorname{Proj}_{\mathcal{H}}(f)$ for $f \in L^2(\Lambda)$

b) $\displaystyle\int_\Lambda K(x, x)\, dx = N$

c) $\displaystyle\int_\Lambda K(x, y) K(y, z)\, dy = K(x, z)$

(Proof is straightforward.)

Observation:

$$\det_{k\times k}\left(K(x_i, x_j)\right) = \det\left(\Phi\,\Phi^*\right)$$

for $k\times N$ matrix $\Phi = \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_N(x_1) \\ \varphi_1(x_2) & & \varphi_N(x_2) \\ \vdots & & \vdots \\ \varphi_1(x_k) & \cdots & \varphi_N(x_k) \end{pmatrix}$

<u>Cor</u>: For $k > N$, $\det_{k\times k}\left(K(x_i, x_j)\right) = 0$, for K as in $(*)$.

<u>Proof</u>: $\Phi\,\Phi^*$ is $k\times k$ but has rank only $N < k$. $\square$

Lemma (Gaudin): For $k+1 \le N$, for K as in (*)

$$\int_\Lambda \det_{(k+1)\times(k+1)} \left(K(x_i, x_j) \right) dx_{k+1} = (N-k) \det_{k\times k} \left(K(x_i, x_j) \right).$$

Proof: Expand by minors along last column:

$$\det_{[k+1]\times[k+1]} \left(K(x_i, x_j) \right) = \det \left[K(x_i, x_j) \;\middle|\; \begin{matrix} K(x_1, x_{k+1}) \\ K(x_2, x_{k+1}) \\ \vdots \\ K(x_{k+1}, x_{k+1}) \end{matrix} \right]$$

$$= \sum_{\ell=1}^{k+1} K(x_\ell, x_{k+1}) (-1)^{k+1+\ell} \cdot \det_{\hat{\ell}\times[k]} \left(K(x_i, x_j) \right)$$

(where $\hat{\ell}\times[k]$ means $i \in \{1, \dots, k+1\} \setminus \ell$, $j \in \{1, \dots, k\}$)

$$= K(x_{k+1}, x_{k+1}) \det_{[k]\times[k]} (K(x_i, x_j))$$

$$+ \sum_{l=1}^{k} K(x_l, x_{k+1}) (-1)^{k+1-l} \det_{\hat{l}\times[k]} (K(x_i, x_j))$$

By Prop,

$$\int_\Lambda \det_{[k+1]\times[k+1]} (K(x_i, x_j)) \, dx_{k+1} = \underbrace{N \cdot \det_{[k]\times[k]} (K(x_i, x_j))}_{a)}$$

$$\underbrace{- \sum_{l=1}^{k} \det_{[k]\times[k]} (K(x_i, x_j))}_{b)}$$

absorb $K(x_l, x_{k+1})$ above into last row, integrate then permute rows

$$= (N-k) \cdot \det_{k\times k} (K(x_i, x_j)) \quad \square$$

Integrating $N-1$ times, this implies ⑥

$$\int_{\Lambda^N} \det_{N\times N}\left(K(x_i,x_j)\right) d^N x = (N-(N-1))(N-(N-2))\cdots(N-1)\underbrace{\int_{\Lambda} K(x_1,x_1)\,dx_1}_{=N}$$

$$= N!$$

(HW gives another way to see this for $U(N)$)

<u>Cor</u>: For K as in $(*)$,

(i) N points with joint pdf

$$f(x_1,\ldots,x_N) = \frac{1}{N!}\det_{N\times N}\left(K(x_i,x_j)\right)$$

are a determinantal projection process, and

(ii) such a function is always a joint pdf on Λ^N.

<u>Proof</u>: (i) proved last time. For (ii) need only $\int_{\Lambda^N} f \, d^N x = 1$ (just showed), and $f(x) \geq 0$ (follows as $f = \det(\Phi\Phi^*) \geq 0$). $\square$

2. Unitary eigenvalues

Thm: e-angles $\{\vartheta_1, \ldots, \vartheta_N\}$ of $g \in \mathcal{U}(N)$ with $\vartheta_i \in [-\frac{1}{2}, \frac{1}{2})$ (so $e^{i2\pi\vartheta_j}$ e-values) satisfy

$$\mathbb{E} \sum_{\substack{j_1, \ldots, j_k \\ \text{distinct}}} \eta(\vartheta_{j_1}, \ldots, \vartheta_{j_k}) = \int_{[-\frac{1}{2}, \frac{1}{2})^k} \eta(x) \cdot \det_{k \times k}\left(K_N(x_i, x_j)\right) d^k x$$

$$\left(\text{for } K_N(x,y) = \sum_{j=0}^{N-1} e^{i2\pi j(x-y)}\right) \tag{1}$$

$$= \int_{[-\frac{1}{2}, \frac{1}{2})^k} \eta(x) \det_{k \times k}\left(S_N(x_i - x_j)\right) d^N x \tag{2}$$

$$\left(\text{for } S_N(x) = \frac{\sin(\pi N x)}{\sin(\pi x)}\right)$$

Proof: (1) follows from joint pdf proved before.
(Note $\{e^{i2\pi jx}\}$ orthonormal.)

(2) $$K_N(x,y) = \frac{e^{i2\pi Nt}-1}{e^{i2\pi t}-1} = e^{i\pi(N-1)t}\,\frac{e^{i\pi Nt}-e^{-i\pi Nt}}{e^{i\pi t}-e^{-i\pi t}}$$

(for $t = x-y$)

Use multilinearity of det to cancel out $e^{i\pi(N-1)(x_i-x_j)}$ factors.

$\square$