Lecture 18

1. Correlation functions for $Sp(2N)$, $SO(2N)$, $SO(2N+1)$

Last time: correlations for $\mathcal{U}(N)$

Key idea: determinantal projection process

$\{e^{i2\pi j\theta}\}_{j=0}^{N-1}$ orthonormal on $[0,1]$

Notation: $S_N(x) = \dfrac{\sin(\pi N x)}{\sin(\pi x)}$ $\leftarrow$ defined by continuity at $x=0$

Symplectic case

<u>Thm</u>: Consider e-angles $\vartheta_1, \dots, \vartheta_N \in [0,1]$ associated to $g \in Sp(2N)$ (so $e^{\pm i\pi\vartheta_j}$ e-values).

For $k \le N$

$$\mathbb{E} \sum_{\substack{j_1, \dots, j_k \\ \text{distinct}}} \eta(\vartheta_{j_1}, \dots, \vartheta_{j_k}) = \int_{[0,1]^k} \eta(x_1, \dots, x_k) \det_{k \times k} \left[L_N(x_i, x_j) \right] d^k x \qquad (1)$$

for $L_N(x,y) = \sum_{j=1}^{N} 2 \sin(\pi j x) \sin(\pi j y)$

$$= \int_{[0,1]^k} \eta(x) \cdot \det_{k \times k} \left[\frac{1}{2} \left(S_{2N+1}\left(\frac{x_i - x_j}{2} \right) - S_{2N+1}\left(\frac{x_i + x_j}{2} \right) \right) \right] d^k x \qquad (2)$$

<u>Proof</u>: For (1): have joint pdf of $(\vartheta_1, \ldots, \vartheta_N)$ on $[0,1]^N$ is

$$\frac{1}{N!} \det_{N \times N} \left[L_N(x_i, x_j) \right]$$

(by rescaling formula proved before for $[0,\pi]^N$)

Formula (1) for k-th joint intensity follows from last class and observation

$$\varphi_j(x) = \sqrt{2} \sin(\pi j x)$$

orthonormal in $L^2([0,1], dx)$

Shown via calculus!

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For (2): have identity

$$L_N(x,y) = \frac{1}{2}\left(S_{2N+1}\left(\frac{x-y}{2}\right) - S_{2N+1}\left(\frac{x+y}{2}\right)\right)$$

use geometric series

$\square$

<u>Cor</u>:

$$\mathbb{E}\sum_j \eta(\vartheta_j) = \int_0^1 \eta(x)\left(\frac{1}{2}\left(2N+1 - S_{2N+1}(x)\right)\right)dx$$

e-values less dense at ± 1 of unit circle

rescaling of 1^{st} intensity claim before

Recall how we started for $Sp(2N)$:

joint pdf was

$$\prod \frac{1}{N!} \det_{N \times N} \left(P_{j-1}(2\cos\theta_k) \cdot \sin\theta_k \right)$$

for any $j-1$ degree monic polys P_{j-1}

In retrospect - got this result for joint integration by choosing P_{j-1} so $P_{j-1}(2\cos\theta)\sin\theta$ were orthogonal.

$\rightsquigarrow$ theory of orthogonal polynomials

Orthogonal case

Thm: Consider e-angles $\vartheta_1, \dots, \vartheta_N \in [0,1]$ associated to $g \in SO(2N)$ ($e^{\pm i\pi\vartheta_j}$ e-values).

For $k \leq N$

$$\mathbb{E} \sum_{\text{distinct}} \eta(\vartheta_{j_1}, \dots, \vartheta_{j_k}) = \int_{[0,1]^k} \eta(x) \cdot \det_{k \times k} \left[\frac{1}{2} \left(S_{2N-1}\left(\frac{x_i - x_j}{2}\right) + S_{2N-1}\left(\frac{x_i + x_j}{2}\right) \right) \right] d^k x$$

For e-angles $\vartheta_1', \ldots, \vartheta_N' \in [0,1]$ associated (1)

to $g \in SO(2N+1)$ ($\{1, e^{\pm i\pi\vartheta_j}\}$ e-values)

$$\mathbb{E} \sum_{\text{distinct}} \eta(\vartheta_{j_1}', \ldots, \vartheta_{j_k}') = \int_{[0,1]^k} \eta \cdot \det_{k \times k} \left[\frac{1}{2} \left(S_{2N}\left(\frac{x_i - x_j}{2}\right) - S_{2N}\left(\frac{x_i + x_j}{2}\right) \right) \right] d^k x$$

<u>Cor</u>: 1^{st} intensity measure for $SO(2N)$:

$$\mathbb{E}\sum \eta(\vartheta_j) = \int_0^1 \eta(x)\left(\frac{1}{2}\left(2N-1 + S_{2N-1}(x)\right)\right)dx$$

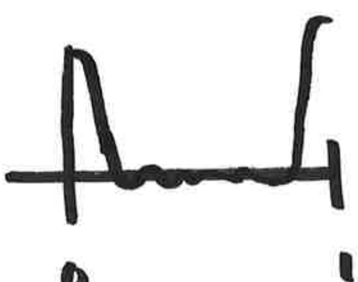
← e-values more dense at ± 1 on unit circle

for $SO(2N+1)$

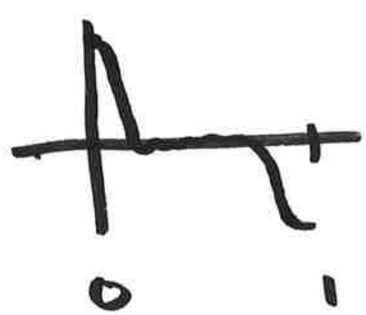
$$\mathbb{E}\sum \eta(\vartheta_j) = \int_0^1 \eta(x)\left(\frac{1}{2}\left[2N - S_{2N}(x)\right]\right)dx$$

← e-values less dense near 1, (except e-value forced at 1) more dense at -1.

$S_{2N-1}(x)$



$S_{2N}(x)$



2. Limiting correlations

Notation

$$S(x) = \begin{cases} \dfrac{\sin \pi x}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Note:

$$\lim_{N \to \infty} \frac{1}{N} S_N\left(\frac{x}{N}\right) = \lim \frac{\sin \pi x}{N \cdot \sin\left(\frac{\pi x}{N}\right)} = \frac{\sin \pi x}{\pi x} = S(x)$$

To consider: e-angles $\{\vartheta_1, \ldots, \vartheta_N\} \in [-\frac{1}{2}, \frac{1}{2})$

of $g \in \mathcal{U}(N)$, rescaled as

$$\{\xi_1, \ldots, \xi_N\} = \{N\vartheta_1, \ldots, N\vartheta_N\} \in [-\tfrac{N}{2}, \tfrac{N}{2})$$

1-level density $= 1$.

Thm: For $\{\xi_i\}$ rescaled e-angles of $\mathcal{U}(N)$, and $\eta \in C_c(\mathbb{R}^k)$,

$$\lim_{N\to\infty} \mathbb{E} \sum_{\substack{j_1, \ldots, j_k \\ \text{distinct}}} \eta(\xi_{j_1}, \ldots, \xi_{j_k}) = \int_{\mathbb{R}^k} \eta(y) \cdot \det_{k\times k}\left(S(y_i - y_j)\right) d^k y$$

Proof :

$$\text{LHS} = \lim_{N\to\infty} \int_{[-\frac{1}{2},\frac{1}{2})^k} \eta(Nx_1, \dots, Nx_k) \det_{k\times k}\left(S_N(x_i - x_j)\right) d^k x$$

$$= \lim \int_{[-\frac{N}{2},\frac{N}{2})^k} \eta(y) \det_{k\times k}\left(S_N\left(\frac{y_i - y_j}{N}\right)\right) \frac{d^k y}{N^k}$$

$$= \lim \int_{\mathbb{R}^k} \eta(y) \cdot \mathbb{1}_{[-\frac{N}{2},\frac{N}{2})^k}(y) \det_{k\times k}\left(\frac{1}{N} S_N\left(\frac{y_i - y_j}{N}\right)\right) d^k y$$

$$= \int_{\mathbb{R}^k} \eta(y) \det_{k\times k}\left(S(y_i - y_j)\right) d^k y. \qquad \square$$