Lecture 19

1. Kernels and determinantal point processes

$\Lambda = [\alpha, \beta]$. Recall $K(x,y) \in L^2(\Lambda \times \Lambda)$ defines bounded linear op. $\underline{K} : L^2(\Lambda) \to L^2(\Lambda)$ by

$$(\underline{K} f)(x) = \int_\Lambda K(x,y) f(y)\, dy.$$

__Def__: If $\underline{K}$ is

- Hermitian $\left(K(x,y) = \overline{K(y,x)} \right)$
- Non-negative definite $\left(\det_{k \times k} \left(K(x_i, x_j) \right) \geq 0 \quad \forall x_1, \dots, x_k \in \Lambda \quad \forall k \right)$
- Of finite rank ($\mathrm{Im}\, K \subset L^2(\Lambda)$ is finite dim.)

then we say $\underline{K}$ (or K) is a *finite admissible kernel*. $\left(\begin{matrix} \text{FA} \\ \text{for short} \end{matrix} \right)$

<u>Fact from linear algebra I</u>: If $\underline{K}$ is FA (of rank N), then

$$K(x,y) = \sum_{\ell=1}^{N} \lambda_\ell \varphi_\ell(x) \overline{\varphi_\ell(y)}$$

for $\{\varphi_\ell\}$ orthonormal in $L^2(\Lambda)$ and $\lambda_\ell \geq 0 \ \forall \ell$

(spectral thm applied to $\underline{K}$)

<u>Fact from linear algebra II</u>: If $\underline{K}$ is FA and $D \subset \Lambda$ and $\underline{K}_D$ is the <u>restriction</u> of $\underline{K}$ to D (i.e. the operator $\underline{K}_D : L^2(D) \to L^2(D)$ defined by

$$(K_D f)(x) = \int_D K(x,y) f(y)\,dy$$

$\leftarrow$ if π_D is the projection onto functions supported on D, $\underline{K}_D = \pi_D \underline{K} \pi_D$)

<u>then $\underline{K}_D$ is also FA.</u>

(Finite rank is only nontrivial part: note $\operatorname{rank}(\pi_D \underline{K} \pi_D) \leq \operatorname{rank}(\underline{K})$.)

Def: A point process ξ is *simple* if for any single point $a \in \mathbb{R}$, the event $\#_{\{a\}}(\xi) \geq 2$ does not occur.

Def: A point process ξ is said to be *determinantal with FA Kernel* $\mathbb{K}$ if its k-th joint intensities are

$$\mathbb{E} \sum_{\substack{j_1, \ldots, j_k \\ \text{distinct}}} \eta(\xi_{j_1}, \ldots, \xi_{j_k}) = \int_{\Lambda^k} \eta(x) \cdot \det_{k \times k} \left(K(x_i, x_j) \right) d^k x$$

for FA Kernel K. (Det. point process = DPP)

2. Some facts about DPP with FA Kernels

Prop: If ξ is a DPP with FA Kernel, of rank N, then $\#_\Lambda(\xi) \leq N$ almost surely.

Proof: For an interval D,

$$\#_D(\#_D - 1) \cdots (\#_D - (k-1)) = \sum_{\substack{j_1, \dots, j_k \\ \text{distinct}}} \mathbb{1}_D(\xi_{j_1}) \cdot \mathbb{1}_D(\xi_{j_2}) \cdots \mathbb{1}_D(\xi_{j_k})$$

(for general point processes)

so

$$\mathbb{E}\, \#_\Lambda(\#_\Lambda - 1) \cdots (\#_\Lambda - (k-1)) = \int_{\Lambda^k} \det_{k \times k}\big(K(x_i, x_j)\big)\, d^k x \qquad (*)$$

But

$$\left(K(x_i, x_j)\right)_{i,j=1}^{k} = \underbrace{\begin{pmatrix} \lambda_1 \varphi_1(x_1) & \lambda_2 \varphi_2(x_1) & \cdots & \lambda_N \varphi_N(x_1) \\ \lambda_1 \varphi_1(x_2) & \lambda_2 \varphi_2(x_2) & \cdots & \\ \vdots & & & \\ \lambda_1 \varphi_1(x_k) & \cdots & - & \lambda_N \varphi_N(x_k) \end{pmatrix}}_{A_\lambda(x)} \underbrace{\begin{pmatrix} \overline{\varphi_1(x_1)} & \cdots & \overline{\varphi_1(x_k)} \\ \vdots & & \vdots \\ \overline{\varphi_N(x_1)} & \cdots & \overline{\varphi_N(x_k)} \end{pmatrix}}_{B(x)}$$

$\leq \operatorname{rank}(N)$

so

$$\det_{k \times k}\left(K(x_i, x_j)\right) = 0 \quad \text{for } k > N,$$

and $(*) = 0$ for $k = N+1$

$\Rightarrow \#_\Lambda = 0, 1, \ldots, \text{or } N$ a.s. $\square$

Cor: If ξ and ξ' are DPP with FA kernel with joint intensities that coincide, then ξ and ξ' have the same distribution (i.e. $\mathbb{P}(C_{n_1}^{B_1} \cap C_{n_2}^{B_2} \cap \dots \cap C_{n_k}^{B_k})$ is the same for both ξ and ξ'.). 6

Proof: Discussed joint intensities
$\Rightarrow$ distribution
if $\#_D$ has exponential tails,
But $\#_D$ is bounded. "$\square$"

3. Projection processes are building blocks

Thm: Let ξ be a DPP with FA Kernel

$$K(x,y) = \sum_{l=1}^{N} \lambda_l \varphi_l(x) \overline{\varphi_l(y)} \quad \text{with } \lambda_l \in [0,1] \ \forall l$$

Let $I_1, \dots, I_N$ be Bernoulli rv with $\mathbb{P}(I_l = 1) = \lambda_l$. Form a point process ξ' by first sampling $I_1, \dots, I_N$, and then sampling from the projection process with kernel

$$K_I(x,y) = \sum_{l=1}^{N} I_l \varphi_l(x) \overline{\varphi_l(y)}.$$

ξ and ξ' have the same dist. (proof next time)