Lecture 2

1. The classical compact groups

The classical compact groups

Def: $O(N) = \{ g \in Mat_{N\times N}(\mathbb{R}) : g^t g = I_N \}$

orthogonal matrices

$U(N) = \{ g \in Mat_{N \times N}(\mathbb{C}) : g^* g = I_N \}$

unitary

$Sp(2N) = \{ g \in Mat_{2N\times 2N}(\mathbb{C}) : g \in U(2N),$
$g^t J_{2N} g = J_{2N} \}$

(compact) symplectic

for $J_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$

Remarks: • $Sp(2N+1)$ is not defined. (2)

• Different authors use different notations for symplectic matrices.

Prop: $g \in O(N)$ iff all N columns are orthonormal elements of $\mathbb{R}^N$

$g \in U(N)$ iff " . . . " of $\mathbb{C}^N$

(just examine def.)

<u>Prop</u>: $O(N) = \{ g : \mathbb{R}^N \to \mathbb{R}^N :$
$\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^N \}$

(for $\langle v, w \rangle = v \cdot w = \sum v_i w_i$)

$U(N) = \{ g : \mathbb{C}^N \to \mathbb{C}^N : \langle gw, gw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{C}^N \}$

(for $\langle v, w \rangle = v \cdot \overline{w} = \sum v_i \overline{w}_i$)

$Sp(2N) = \{ g : \mathbb{C}^{2N} \to \mathbb{C}^{2N} : \langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{C}^{2N}$
and $\omega(gv, gw) = \omega(v, w)$ for all $v, w \in \mathbb{C}^{2N} \}$

(for $\langle v, w \rangle = v \cdot \overline{w}$ and $\omega(v, w) = v \cdot Jw = v_1 \cdot w_{N+1} + \cdots + v_N w_{2N} - v_{N+1} w_1 - \cdots - v_{2N} w_N$)

Proof: $g \in U(N) \Leftrightarrow \langle gv, gw \rangle = \langle v, g^* g w \rangle$

$= \langle v, w \rangle \quad \forall v, w$

$\Leftrightarrow g$ leaves inner product invariant

$g \in O(N)$ has same proof.

$g \in Sp(2N) \Longleftrightarrow \begin{array}{l} g \in U(2N) \\ g^t J g = J \end{array} \Longleftrightarrow \begin{array}{l} \langle gu, gv \rangle = \langle u, v \rangle \\ \forall u, w \in \mathbb{C}^{2N} \\ u \cdot (g^t J g v) = u \cdot Jv \\ \Updownarrow \\ gu \cdot Jgv = u \cdot Jv \end{array}$

$\square$

<u>Thm</u>: $O(N)$, $U(N)$, and $Sp(2N)$ are all compact groups.

<u>Proof</u>: Groups: For $U(N)$:
(i) $I \in U(N)$, (ii) g_1, g_2 preserve $\langle \cdot, \cdot \rangle$
$\Rightarrow g_1 g_2$ preserves $\langle \cdot, \cdot \rangle$.
(iii) g preserves $\langle \cdot, \cdot \rangle$
$\Rightarrow g^{-1}$ preserves $\langle \cdot, \cdot \rangle$

Same for $O(N)$, same for $Sp(2N)$ except also note preservation of ω under group action.

Compact: (Inherit topology from $Mat_{N\times N}(\mathbb{R})$ or $Mat_{N\times N}(\mathbb{C})$)

For $O(N)$, $U(N)$ all column vectors are orthonormal.

$\Rightarrow O(N) \underset{\text{compact}}{\subset} Mat_{N\times N}(\mathbb{R})$,

$U(N) \underset{\text{compact}}{\subset} Mat_{N\times N}(\mathbb{C})$,

And $Sp(2N) \subset U(2N)$ so compact in $Mat_{2N\times 2N}(\mathbb{C})$.

(Closure of these sets is clear also.) $\square$

2. What's going on with $Sp(2N)$?

Recall quaternions: $\mathbb{H} = \{a + ib + jc + kd : a, b, c, d \in \mathbb{R}\}$
with $i^2 = j^2 = k^2 = -1$ and $ijk = -1$

Frobenius thm: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only finite dim. associate division algebras over the real numbers.

Def ($\mathbb{H}$ conjugation):

$$\overline{a + ib + jc + kd} = a - ib - jc - kd$$

Recall: Matrix realization of $\mathbb{C}$:

$$\mathbb{C} \cong \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Matrix realization of $\mathbb{H}$:

$$\mathbb{H} \cong \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + di\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Can thus represent

$$G = A + iB + jC + kD \in Mat_{N\times N}(\mathbb{H})$$

(with $A, B, C, D \in Mat_{N\times N}(\mathbb{R})$)

by the complex matrix

$$g = \begin{pmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{pmatrix} \in Mat_{2N\times 2N}(\mathbb{C})$$

with matrix mult. preserved.

Now note g is of the form:

$$g = \begin{pmatrix} P & Q \\ -\bar{Q} & \bar{P} \end{pmatrix} \Leftrightarrow g = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \text{ with } \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \bar{P} & \bar{Q} \\ \bar{R} & \bar{S} \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$\Leftrightarrow g \text{ satisfies } Jg = \bar{g}J$$

for $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Thus

$$Mat_{N\times N}(\mathbb{H}) \cong \{ g \in Mat_{2N\times 2N}(\mathbb{C}) : Jg = \bar{g}J \}$$