Lecture 20

1. Projection processes are building blocks (cont.)

Recall the thm we ended with in Lecture 19.

Refer back to factorization of $(K(x_i, x_j))_{i,j=1}^{k}$ on p. 5 of [Notes 19] $= A_\lambda(x) \cdot B(x)$

Recall Cauchy-Binet formula: For $n \geq k$,

$A \in Mat_{k \times n}$, $B \in Mat_{n \times k}$

$$\underset{k \times k}{\det}(AB) = \sum_{S \in \binom{[n]}{k}} \det(A_{[k] \times S}) \cdot \det(B_{S \times [k]})$$

(HW status with roles of A, B reversed)

<u>Proof</u>: As ξ and ξ' have at most N points, need only see that they have the same joint intensities.

$$\mathbb{E}\sum_{\text{distinct}} \eta(\xi'_{j_1},\dots,\xi'_{j_n}) = \mathbb{E}_I \int_{\Lambda^k} \eta\cdot \det_{k\times k}\left(K_I(x_i,x_j)\right)d^k x$$

Cauchy-Binet $\longrightarrow$
$$= \mathbb{E}_I \int_{\Lambda^k} \eta\cdot \sum_{S\in\binom{[N]}{k}} \det\left([A_I(x)]_{[k]\times S}\right) \cdot \det\left([B(x)]_{S\times[k]}\right) \cdot d^k x$$

$$= \int_{\Lambda^k} \eta\cdot \sum_{S\in\binom{[N]}{k}} \det\left([A_\lambda(x)]_{[k]\times S}\right) \cdot \det\left([B(x)]_{S\times[k]}\right) \cdot d^k x$$

Cauchy-Binet $\longrightarrow \quad = \int_{\Lambda^k} \eta \cdot \det_{k\times k}\left(K(x_i, x_j)\right) d^k x$

$\square$

<u>Thm</u> (Macchi) Let $K \in L^2(\Lambda \times \Lambda)$ be a FA Kernel. The measures

$$\rho_k(x_1, \dots, x_k)\, d^k x = \det_{k\times k}\left(K(x_i, x_j)\right) d^k x$$

are joint intensities of a point process iff the spectrum $\lambda_\ell \in [0,1]$ for all ℓ.

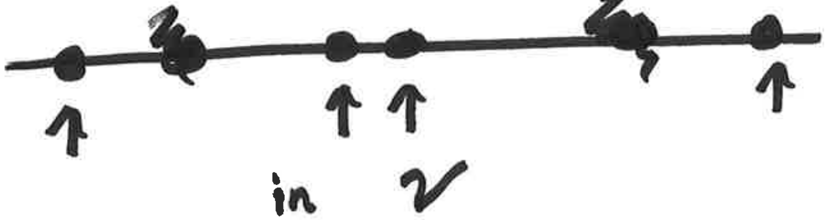
(Recall $K(x,y) = \sum_{\ell \geq 1}^{N} \lambda_\ell \varphi_\ell(x) \overline{\varphi_\ell(y)}$ by linear algebra.)

<u>Proof</u>: $\lambda_\ell \in [0,1] \Rightarrow$ point process:

use construction just given!

Point process $\Rightarrow \lambda_\ell \in [0,1]$: • Suppose there is such a point process ξ with some $\lambda_\ell \notin [0,1]$. All $\lambda_\ell \geq 0$ (for any FA kernel — linear algebra fact I), so must have some $\lambda_\ell > 1$.

- Let $\lambda = \max \lambda_\ell$
- Form point process ν by sampling configs of ξ, then independently deleting individual points with prob. $1 - \frac{1}{\lambda}$.



Claim: ν has joint intensities

$$\det_{k\times k}\left(\frac{1}{\lambda}K(x_i,x_j)\right)d^kx$$

<u>Why?</u>

$$\mathbb{E}\sum_{\text{distinct}}\eta(\nu_{j_1},\dots,\nu_{j_k}) = \mathbb{E}\left(\frac{1}{\lambda}\right)^k\sum_{\text{distinct}}\eta(\xi_{j_1},\dots,\xi_{j_k})$$

(as prob. $\frac{1}{\lambda}$ that each ξ_{j_i} survives)

$$= \frac{1}{\lambda^k}\int_{\Lambda^k}\eta\cdot\det_{k\times k}\left(K(x_i,x_j)\right)d^kx$$

✓

Now note:

[ξ has finitely many points, so positive prob. all are deleted;

$$\mathbb{P}(\#_\Lambda(\nu) = 0) > 0$$

[But from new kernel, $\frac{1}{\lambda}K(x,y)$ has an e-value $= 1$ (and all other e-values ≤ 1)

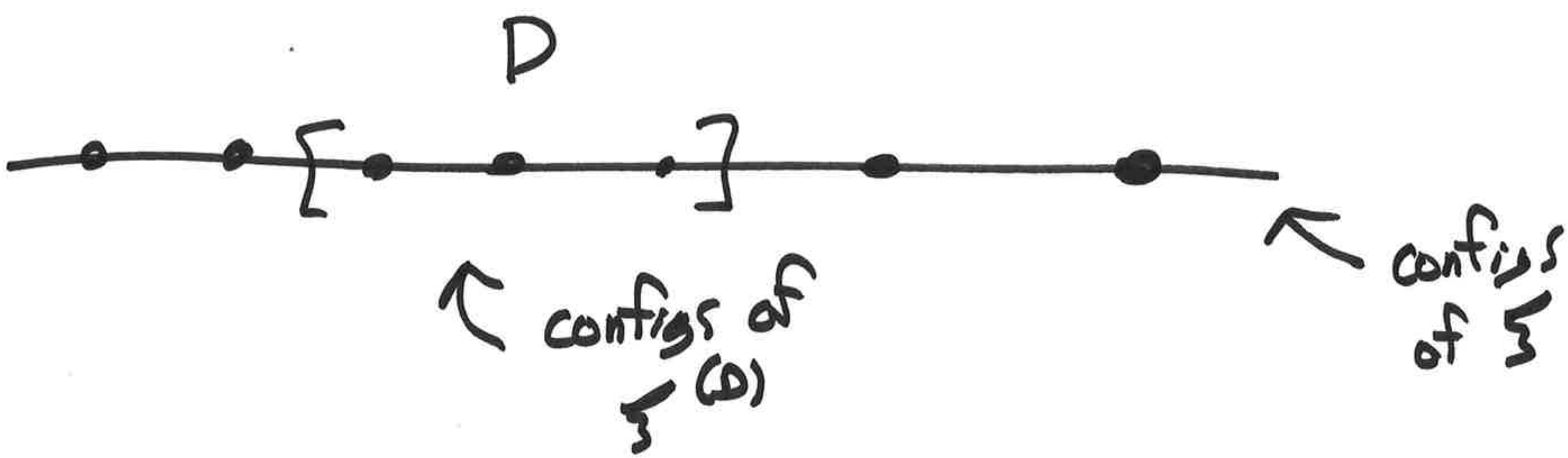
$\Rightarrow$ In projection process construction of ν, some Bernoulli $N = 1$ a.s.

$$\Rightarrow \mathbb{P}(\#_\nu(\nu) \geq 1) = 1$$

Contradiction! $\square$

2. A central limit theorem for DPP (Part I)

Observation: Take ξ a DPP with FA Kernel on $\Lambda = [\alpha, \beta]$. For any interval* $D \subset [\alpha, \beta]$, let $\xi^{(D)}$ be the point process of points from ξ lying in D



(1) $\xi^{(D)}$ is a DPP with FA Kernel

(As $\xi^{(D)}$ has joint intensities

$$\mathbb{1}_{D^k}(x) \det_{k \times k}\left(K(x_i, x_j)\right) = \det_{k \times k}\left(K^{(D)}(x_i, x_j)\right)$$

for $K^{(D)}(x,y) = \mathbb{1}_D(x)\, K(x,y)\, \mathbb{1}_D(y)$.)

FA Kernel by linear algebra fact II

(2) $\#_D(\xi) = I_1 + \cdots + I_N$ where

$I_\ell \sim \text{Bernoulli}(\lambda_i)$ independent,

with $\lambda_1, \ldots, \lambda_N$ the e-values of $\underline{K}^{(D)}$,

with $\lambda_\ell \in [0,1]$.

(As $\#_D(\xi) = \#(\xi^{(D)})$, and $\xi^{(D)}$ is got from sampling projection processes of rank $I_1 + \cdots + I_N$)

* could replace an interval with a Borel set.