Lecture 28

1. Schur functions in e-values

For $g \in \mathcal{U}(N)$, let $\{e(\vartheta_1), \ldots, e(\vartheta_N)\}$ be the e-values. Shorthand: $\omega_i = e(\vartheta_i)$

Notation: $s_\lambda(g) := s_\lambda(\omega_1, \ldots, \omega_N)$.

Def: (Inner product on $\mathcal{U}(N)$) For $\varphi(g) = \varphi(\omega_1, \ldots, \omega_N)$ and $\psi(g) = \psi(\omega_1, \ldots, \omega_N)$ sym. functions in e-values of $g \in \mathcal{U}(N)$,

$$\langle \varphi, \psi \rangle_{\mathcal{U}(N)} := \int_{\mathcal{U}(N)} \varphi(g) \overline{\psi(g)} \, dg .$$

Observe: as $a_\delta(\omega_1, \dots, \omega_N) = \prod_{i<j} (\omega_i - \omega_j)$, ②

this is

$$\langle \varphi, \psi \rangle_{\mathcal{U}(N)} = \frac{1}{N!} \int_{T^N} \varphi \cdot \overline{\psi} \, |a_\delta|^2 \, d\vartheta_1 \cdots d\vartheta_N$$

by Weyl integral formula

<u>Prop</u>:

$$\langle s_\lambda, s_\mu \rangle_{\mathcal{U}(N)} = \mathbb{1}_{\lambda = \mu} \cdot \mathbb{1}_{\ell(\lambda) \le N, \ell(\mu) \le N}$$

Proof: If $\ell(\lambda) > N$ or $\ell(\mu) > N$, have (3)

$s_\lambda(g) = 0$ or $s_\mu(g) = 0$, so Prop is

clear in this case.

Preliminary: if $\ell(\lambda), \ell(\mu) \leq N$ and λ, μ are strictly decreasing ($\lambda_1 > \lambda_2 > \cdots$ and $\mu_1 > \mu_2 > \cdots$)

$$\int_{T^N} a_\lambda(\omega_1, \ldots, \omega_N) \overline{a_\mu(\omega_1, \ldots, \omega_N)}\, d^N\vartheta = \int_{T^N} \det_{N\times N}\left(\omega_i^{\lambda_j}\right) \det_{N\times N}\left(\omega_i^{-\mu_j}\right) d^N\vartheta$$

$$= \int_{T^N} \sum_{\sigma, \rho \in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho) \prod_{i=1}^{N} \omega_i^{\lambda_{\sigma(i)} - \mu_{\rho(i)}}\, d^N\vartheta$$

same as HW2 $\longrightarrow$ $= \mathbb{1}_{\lambda=\mu} \sum_{\sigma=\rho\in S_N} \int_{T^N} 1 \cdot d^N\vartheta = N!\, \mathbb{1}_{\lambda=\mu}$

Thus if $\ell(\lambda), \ell(\mu) \leq N$, because $\lambda+\delta, \mu+\delta$ will be strictly decreasing,

$$\langle s_\lambda, s_\mu \rangle_{U(N)} = \frac{1}{N!} \int_{T^N} \left(\frac{a_{\lambda+\delta}}{a_\delta}\right) \overline{\left(\frac{a_{\mu+\delta}}{a_\delta}\right)} |a_\delta|^2 d^N\vartheta$$

$$= \frac{1}{N!} \int_{T^N} a_{\lambda+\delta} \overline{a_{\mu+\delta}}\, d^N\vartheta$$

$$= \mathbb{1}_{\lambda=\mu}. \qquad \square$$

2. Power sums in e-values

As formal power series

$\{p_\lambda : \lambda \vdash n\}$ and $\{s_\lambda : \lambda \vdash n\}$ are a basis for Λ^n

Hence for any $\lambda \vdash n$,

(*) $$p_\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu s_\mu \quad \text{for some } \chi^\lambda_\mu \in \mathbb{C}.$$

(*) is also true for $p_\lambda(\omega_1, \dots, \omega_N)$, but some $s_\mu(\omega_1, \dots, \omega_N) = 0$:

$$p_\lambda(\omega_1, \dots, \omega_N) = \sum_{\substack{\mu \vdash n \\ \ell(\mu) \le N}} \chi^\lambda_\mu s_\mu(\omega_1, \dots, \omega_N).$$

But note: if $n \leq N$, then $\ell(\mu) \leq N$ imposes no restriction

Thm (Diaconis–Shahshahani): If $\lambda, \lambda' \vdash n$, with $n \leq N$, then

$$\langle p_\lambda, p_{\lambda'} \rangle_{\mathcal{U}(N)} = \mathbb{1}_{\lambda = \lambda'} \cdot z_\lambda$$

Proof:

$$\langle p_\lambda, p_{\lambda'} \rangle_{\mathcal{U}(N)} = \sum_{\mu, \mu' \vdash n} \chi^\lambda_\mu \overline{\chi^{\lambda'}_{\mu'}} \langle s_\mu, s_{\mu'} \rangle_{\mathcal{U}(N)}$$

$$= \sum_{\mu, \mu' \vdash n} \chi^\lambda_\mu \overline{\chi^{\lambda'}_{\mu'}} \underbrace{\mathbb{1}_{\mu = \mu'}}_{\langle s_\mu, s_{\mu'} \rangle} \quad \left(\text{as } \ell(\mu), \ell(\mu') \leq N\right)$$

$$= \langle p_\lambda, p_{\lambda'} \rangle$$

$$= \mathbb{1}_{\lambda=\lambda'} \cdot z_\lambda . \qquad \square$$

False if $n > N$! Would have:

$$\langle p_\lambda, p_\lambda \rangle_{U(N)} = \sum_{\substack{\mu \vdash n \\ \ell(\mu) \le N}} |\chi^\lambda_\mu|^2 < \langle p_\lambda, p_\lambda \rangle$$

at least if $|\chi^\lambda_\mu| > 0$ for some $\ell(\mu) > N$.

3. The CLT for traces

Observe $\displaystyle\prod_{j=1}^{J}(\mathrm{Tr}(g^j))^{a_j} = P_\lambda(g)$

for $\lambda = (1^{a_1} 2^{a_2} \cdots J^{a_j})$

$|\lambda| = \displaystyle\sum_{j=1}^{J} j a_j$

$Z_\lambda = 1^{a_1} a_1!\, 2^{a_2} a_2! \cdots J^{a_j} a_j!$

<u>Thm</u> (Diaconis-ShahShahani restated):

For $\sum j a_j \leq N$, $\sum j b_j \leq N$,

$$\int_{U(N)} \prod_{j=1}^{J} \mathrm{Tr}(g^j)^{a_j} \overline{\prod_{j=1}^{J}\left(\mathrm{Tr}(g^j)^{b_j}\right)}$$

$$= \mathbb{1}_{\vec{a}=\vec{b}} \cdot \prod_{j=1}^{J} j^{a_j} \cdot a_j!$$