Lecture 29

1. Traces and complex gaussians

<u>Def</u>: A complex random variable Z is *subexponential* if $\mathbb{P}(|Z|>t) \leq Ae^{-ct}$ for constants $A, c > 0$, and all $t \geq 0$.

<u>Fact from probability</u>: Let $Z_1, \ldots, Z_J$ be subexp. complex random var, and $(Z_1^{(n)}, \ldots, Z_J^{(n)})$ be a sequence of random complex vectors.

If

(over)

$$\lim_{n\to\infty} \mathbb{E} \prod_{j=1}^{J} (Z_j^{(n)})^{a_j} \overline{(Z_j^{(n)})}^{b_j} = \mathbb{E} \prod_{j=1}^{J} (Z_j)^{a_j} \overline{(Z_j)}^{b_j} \quad (2)$$

for $a, b \in (\mathbb{N}_{\geq 0})^J$, then as $n \to \infty$

$$(Z_1^{(n)}, \dots, Z_J^{(n)}) \xrightarrow{dist} (Z_1, \dots, Z_J)$$

(Same as moment method for reals, but need conjugates for complex random var.)

(Ex: for real random var. $S^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$, $X_i = \pm 1$ with equal prob.

then can show $\mathbb{E}(S^{(n)})^k \to \mathbb{E} G^k$ for G gaussian,

and this implies $S^{(n)} \xrightarrow{dist.} G$)

Fact from probability / calculus: If $G \sim N_{\mathbb{C}}(0,1)$, then for $a, b \in \mathbb{N}_{\geq 0}$,

$$\mathbb{E}\, G^a \bar{G}^b = \frac{1}{\pi} \int_{\mathbb{C}} z^a \bar{z}^b e^{-|z|^2} \, dx\, dy$$
$$= \mathbb{1}_{a=b} \cdot a!$$

Thus: if $G_1, \ldots, G_J$ are iid $\sim N_{\mathbb{C}}(0,1)$, then

$$\mathbb{E} \prod_{j=1}^{J} (\sqrt{j}\, G_j)^{a_j} \overline{(\sqrt{j}\, G_j)}^{b_j} = \prod_{j=1}^{J} j^{(a_j+b_j)/2}\, \mathbb{E}\, G_j^{a_j} \bar{G}_j^{b_j}$$
$$= \mathbb{1}_{\vec{a}=\vec{b}} \cdot \prod_{j=1}^{J} j^{a_j} a_j!$$

Same as $\mathrm{Tr}(g^j)$! (for large enough N)

Hence as $N \to \infty$

$$\left[\text{moments of } (\mathrm{Tr}(g), \mathrm{Tr}(g^2), \dots, \mathrm{Tr}(g^J))\right] \to \left[\text{moments of } (G_1, \sqrt{2}G_2, \dots, \sqrt{J}G_J)\right]$$

$G_1, \sqrt{2}G_2, \dots, \sqrt{J}G_J$ subexp., so

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ \text{(D-S): For fixed } J, \text{ as } N \to \infty \\ \quad (\mathrm{Tr}(g), \dots, \mathrm{Tr}(g^J)) \xrightarrow{\text{dist.}} (G_1, \sqrt{2}G_2, \dots, \sqrt{J}G_J) \\ \quad\quad\quad\quad\quad\quad\quad \text{for } g \in \mathcal{U}(N)\end{array}}$$

2. Linear statistics and the strong Szegő' CLT

Observation: if $f(\theta) = \sum_{j=-J}^{J} \hat{f}_j e(j\theta) \leftarrow$ (a "trig. polynomial")

then for $\{\vartheta_1, \ldots, \vartheta_N\}$ be e-angles of $g \in \mathcal{U}(N)$,

$$\sum_{n=1}^{N} f(\vartheta_N) = \sum_j \hat{f}_j \sum_{n=1}^{N} \left(e^{i2\pi\vartheta_n}\right)^j = \sum_j \hat{f}_j \operatorname{Tr}(g^j)$$

$$= N\hat{f}_0 + \sum_{j=1}^{J} \hat{f}_j \operatorname{Tr}(g^j) + \hat{f}_{-j} \overline{\operatorname{Tr}(g^j)}$$

Suppose $f: \mathbb{T} \to \mathbb{R}$, then $\hat{f}_j = \int_{\mathbb{T}} e^{-i2\pi j\theta} f(\theta) = \overline{\hat{f}_{-j}}$.

Also $\hat{f}_0 = \int_{\mathbb{T}} f(\theta)\, d\theta$.

so

$$\sum_{n=1}^{N} f(\vartheta_n) - N\int_{T} f(\theta)\,d\theta = 2\mathcal{R}\left(\sum_{j=1}^{J} \hat{f}_j \, \mathrm{Tr}(g^{j})\right)$$

$$\xrightarrow{\text{dist.}} 2\mathcal{R}\left(\sum_{j=1}^{J} \hat{f}_j \sqrt{j}\, G_j\right)$$

$$\sim 2\mathcal{R}\left(N_{\mathbb{C}}\left(0, \sum_{j=1}^{J} j\,|\hat{f}_j|^2\right)\right)$$

$$\sim N_{\mathbb{R}}\left(0, 2\sum_{j=1}^{J} j\,|\hat{f}_j|^2\right)$$

$$= N_{\mathbb{R}}\left(0, \sum_{j=-J}^{J} |j|\,|\hat{f}_j|^2\right)$$

<u>Thm</u>: $$\sum_{n=1}^{N} f(\vartheta_n) - N\int_{\mathbb{T}} f \xrightarrow{dist.} N_{\mathbb{R}}\left(0, \sum_j |j||\hat{f}_j|^2\right).$$

1) This thm is true for any $f: \mathbb{T} \to \mathbb{R}$ as long as $\sum_{j=-\infty}^{\infty} |j||\hat{f}_j|^2 < +\infty$. Known as the "Strong Szegő thm" - 6 proofs in B. Simons OPUC v. 1

2) These sorts of ideas can be applied to f which vary with N e.g. $f = \mathbb{1}_{[-\frac{L}{2N}, \frac{L}{2N}]}$ with $L = L(N)$. - See Diaconis & Evans or Meckes Ch. 4.2

3. Connection to Toeplitz determinants

Szegő CLT $\Rightarrow$ $$\int_{\mathcal{U}(N)} \exp\Big(\lambda \sum f(\vartheta_n)\Big)\, dg \sim e^{\lambda \cdot N \int f} \cdot \exp\Big(\frac{\lambda^2}{2} \sum_{j=-\infty}^{\infty} |j|\,|\hat{f}_j|^2\Big)$$

Let $\lambda = 1$, $F(\theta) = e^{f(\theta)}$:

$$\int_{\mathcal{U}(N)} \exp\Big(\sum f(\vartheta_n)\Big)\, dg = \frac{1}{N!} \int_{\mathbb{T}^N} F(\vartheta_1) \cdots F(\vartheta_N) \det_{N\times N}\big(e(j\vartheta_k)\big) \det_{N\times N}\big(e(-j\vartheta_k)\big)\, d^N\vartheta$$

$$= \det_{N\times N}\Big(\int_{\mathbb{T}} F(\theta)\, e((j-k)\theta)\, d\theta \Big)$$

identity of Andreief in HW

$$= \det \begin{pmatrix} \hat{F}_0 & \hat{F}_{-1} & \cdots & \hat{F}_{-(N-1)} \\ \hat{F}_1 & \hat{F}_0 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

Toeplitz det.

9

So Strong Szegö is also about the asymptotics of Toeplitz determinants.