Lecture 3

1. What's going on with Sp(n)? (cont.)

Last class:

$$\mathrm{Mat}_{N\times N}(\mathbb{H}) \cong \{ g \in \mathrm{Mat}_{2N\times 2N}(\mathbb{C}) : Jg = \bar{g} J \}$$

with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

under correspondance

$$A + iB + jC + kD \longleftrightarrow \begin{pmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{pmatrix}$$

<u>Def</u>: (Quaternionic adjoint) For $G \in \mathrm{Mat}_{N\times N}(\mathbb{H})$

$$G^{\circledast} := \overline{G}^{t}.$$

(non-standard notation)

(2)

<u>Observe</u>: If $G = A + iB + jC + kD$

then $G^{\circledast} = A^t - iB^t - jC^t - kD^t$

so

$$G \longmapsto G^{\circledast}$$

Corresponds to

$$g = \begin{pmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{pmatrix} \longmapsto \begin{pmatrix} A^t-iB^t & -C^t-iD^t \\ C^t-iD^t & A^t+iB^t \end{pmatrix} = g^*$$

Hence

$$G^{\circledast} G = I$$

Corresponds to

$$g^* g = I$$

Prop:
$$O(N)=\{G^tG=I : G\in Mat_{N\times N}(\mathbb{R})\}$$
$$U(N)=\{G^*G=I : G\in Mat_{N\times N}(\mathbb{C})\}$$
$$Sp(2N)\cong\{G^{\circledast}G=I : G\in Mat_{N\times N}(\mathbb{H})\}$$

Proof for $Sp(2N)$:

$$\{G^{\circledast}G=I \text{ and } G\in Mat_{N\times N}(\mathbb{H})\}$$

corresponds to

$$\{g^*g=I \text{ and } Jg=\bar{g}J \text{ with } g\in Mat_{2N\times 2N}(\mathbb{C})\}$$

But if $g^*g=I$ then $g^t\bar{g}=I$, so $Jg=\bar{g}J \iff g^tJg=J$.

Hence the above matrices are just $Sp(2N)$. $\square$

Hence: "$G \in Sp(2N)$ if all N columns (of the quaternionic representation) are orthonormal elements of $\mathbb{H}^N$"

2. Haar measure

Def: G is a topological group if G is a group and the set G has Hausdorff topology such that

$$(x,y) \longmapsto xy \quad \text{from } G \times G \to G$$

$$x \longmapsto x^{-1} \quad \text{from } G \to G$$

are continuous.

<u>Prop</u>: $O(N)$, $U(N)$, and $Sp(2N)$ are compact topological groups.

: (Inherit the topology from $Mat_{N\times N}(\mathbb{R}) \cong \mathbb{R}^{N^2}$, $Mat_{N\times N}(\mathbb{C}) \cong \mathbb{C}^{N^2} \cong \mathbb{R}^{2N^2}$, $Mat_{2N\times 2N}(\mathbb{C})$)

Recall: <u>the Borel σ-algebra</u> on a topological space is the σ-algebra generated by open sets.

A <u>Radon measure</u> is a measure on the σ-algebra of Borel sets of a Hausdorff top. space that is finite on compact sets and regular ($m(\mathcal{U}) = \sup\{m(K) : \text{compact } K \subset \mathcal{U}\}$ for $\mathcal{U}$ open, $m(B) = \inf\{m(\mathcal{U}) : \text{open } \mathcal{U} \supset B\}$ for B Borel.)

Not too important!

<u>Def</u>: For G a topological group, a <u>left Haar measure</u> is a non-trivial Radon measure μ satisfies $\mu(hA) = \mu(A)$ for all $h \in G$ and Borel set $A \subseteq G$.

(non-trivial = not identically zero). A <u>right Haar measure</u> ν satisfies $\nu(Ah) = \nu(A)$ for all h, A.

<u>Ex</u>: For any finite group G (with discrete topology) the counting measure $\mu(A) = |A|$ is a left and right Haar measure.

For $G = (\mathbb{R}, +)$ or $G = (\mathbb{R}/\mathbb{Z}, +)$, $d\mu = c \cdot dx$ (for any constant c) is a left and right Haar measure.

If $\mu(G)=1$, we say μ is a (left/right) Haar probability measure.

Thm: If G is a compact topological group, there exists a unique left Haar prob. measure μ, and a unique right Haar prob. measure ν, and moreover $\mu=\nu$.

(Note this is true for locally compact G, except can't force $\mu(G)=\nu(G)=1$, so uniqueness only up to scaling factor, and can have $\mu\neq\nu$.)

(Rough) proof of uniqueness:

Consider left Haar μ and right Haar ν with $\mu(G) = \nu(G) = 1$. For $\varphi \in C(G)$,

$$\int_G \varphi(x)\, d\mu(x) = \int_G \varphi(yx)\, d\mu(x) \qquad \forall y \in G$$

$$= \int_G \left(\int_G \varphi(yx)\, d\mu(x) \right) d\nu(y)$$

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$$= \int_G \left(\int_G \varphi(y)\, d\nu(y) \right) d\mu(x) = \int_G \varphi(y)\, d\nu(y)$$

Why a "rough" proof? Concluding $\mu = \nu$ from this takes Riesz rep thm.

So $\mu = \nu$. But if μ' is another left Haar (9)
measure $\mu' = \nu = \mu$.

Hence at most one left/right measure. $\square$

We won't demonstrate existence for
all compact topological groups, but
we give an <u>explicit</u> construction for
$G = O(N), U(N), Sp(2N)$.