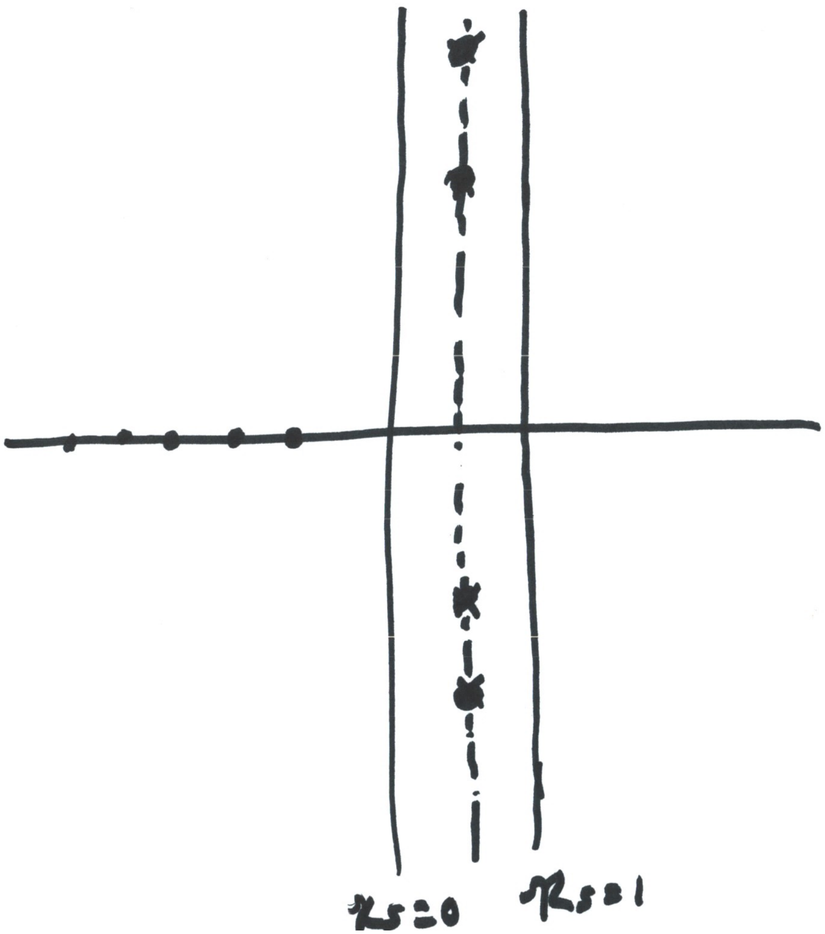
Lecture 30

1. The Riemann Zeta function

<u>Def</u>: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\Re(s) > 1$.



<u>Prop</u> (Riemann, etc.) $\zeta(s)$ is analytic for $\Re s > 1$, and has a unique meromorphic cont. to all of $\mathbb{C}$, with a simple pole at $s = 1$.

- $\zeta(s)$ has trivial zeros at $s = -2, -4, -6, \ldots$, and nontrivial zeros lying entirely in the critical strip $\{s : \Re s \in (0,1)\}$

- Nontrivial zeros have the symmetry that if $\zeta(\sigma+i\gamma)=0$ then $\zeta(\sigma-i\gamma)=0$ also.

- If $N(T) = \#\{\gamma\in[0,T] : \sigma+i\gamma \text{ a nontrivial zero}\}$

then

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$$

$$\sim T\,\frac{\log T}{2\pi}$$

(around height T, zeros have density roughly $\frac{\log T}{2\pi}$)

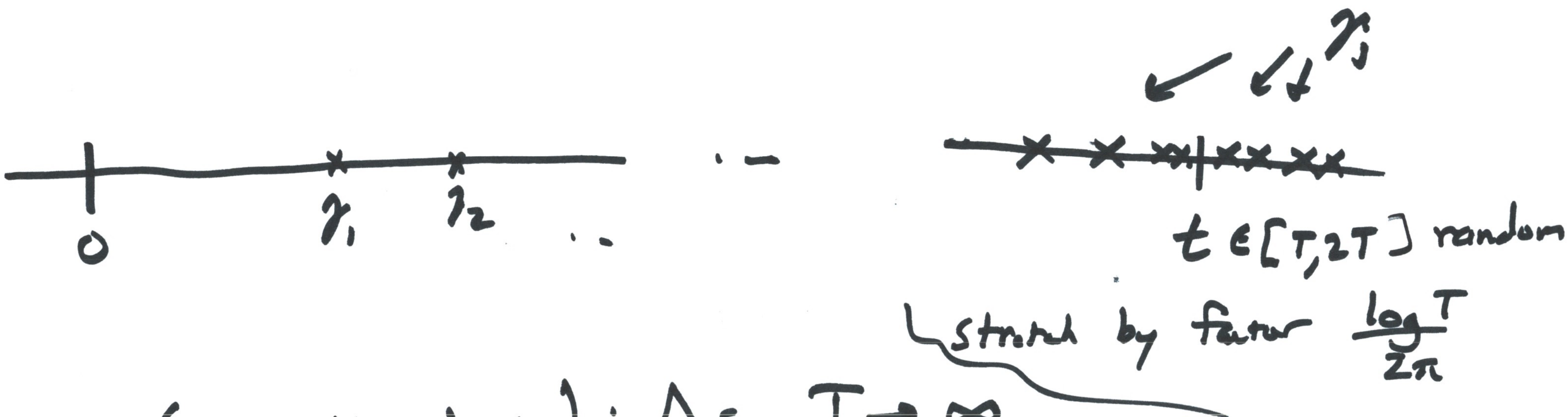
Riemann Hypothesis: All nontrivial zeros $\sigma + i\gamma$ satisfy $\sigma = \frac{1}{2}$.

2. Zeta zero point processes

- Assume RH, so $\{\frac{1}{2} + i\gamma_j\}_{j \in \mathbb{Z}}$ is a complete list of non-trivial zeros, $\ldots \leq \gamma_{-2} \leq \gamma_{-1} < 0 < \gamma_1 \leq \gamma_2 \leq \ldots$
- Take T large, and let τ be a random var, uniformly dist. $[T, 2T]$.

Consider the point process

$$Z_T \quad \text{with configs} \quad \left\{ \frac{\log T}{2\pi}(\gamma_j - t) \right\}$$



<u>Conj</u> (GUE Hypothesis): As $T \to \infty$,

$$Z_T \xrightarrow{\text{dist.}} \begin{pmatrix} \text{sine-Kernel} \\ \\ \text{process} \end{pmatrix}$$

<u>Notation</u>: (Schwartz functions)

$\varphi \in S(\mathbb{R}^k)$ if $\varphi : \mathbb{R}^k \to \mathbb{C}$ and as $|x| \to \infty$, $\varphi(x)$ decays faster than $1/|x|^A$ for any $A > 0$, and we have the same decay for partial derivatives of any order of φ. (e.g.

$$\frac{\partial^3}{\partial x_1^3} \frac{\partial}{\partial x_2} \varphi(x_1, x_2) = O_A\left(\frac{1}{\left((x_1^2 + x_2^2)^{1/2}\right)^A}\right)$$

for large (x_1, x_2).)

GUE Hyp ends up equivalent to:

<u>GUE Hyp (correlation form)</u>: For any $k \geq 1$, for any $\varphi \in S(\mathbb{R}^k)$,

$$\begin{pmatrix}\text{GUE}\\ \text{-Corr}\\ k\end{pmatrix} \qquad \lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \sum_{\substack{j_1,\dots,j_k\\ \text{distinct}}} \varphi\left(\frac{\log T}{2\pi}(\gamma_{j_1}-t), \dots, \frac{\log T}{2\pi}(\gamma_{j_k}-t)\right) dt$$

$$= \int_{\mathbb{R}^k} \varphi(x)\cdot \det_{k\times k}\left[S(x_i-x_j)\right] d^k x$$

$\left(\text{for } S(x) = \dfrac{\sin \pi x}{\pi x}\right)$

Also equivalent to:

<u>GUE Hyp (moment form)</u>: For any $k \geq 1$, and any $\eta \in S(\mathbb{R})$,

(GUE-moment k)

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \left(\sum_j \eta\left(\frac{\log T}{2\pi}(\gamma_j - t)\right) - \int_{-\infty}^{\infty} \eta(x)\,dx \right)^k dt$$

$$= \mathbb{E}\left(\sum_j \eta(\xi_j) - \int_{-\infty}^{\infty} \eta \right)^k$$

where $\{\xi_j\}_{j\in\mathbb{Z}}$ are the configs of the sine-kernel process.

Note: if $\varphi \in C_c(\mathbb{R}^k)$ or $\eta \in C_c(\mathbb{R})$, we have already shown these are equiv. to distributional form of GUE. For $\varphi \in S(\mathbb{R}^k)$, $\eta \in S(\mathbb{R})$ equivalence depends on a bound on 'tails' of linear stats. Not hard, but does require a little work.

<u>Prop</u>: 1-level density (GUE-corr-1 = GUE-moment-1) is true for all test functions $\eta \in S(\mathbb{R})$.

<u>Idea of proof</u>: Just a restatement that density near t is $\approx \frac{\log T}{2\pi}$:

$$\frac{1}{T}\int_T^{2T} \sum_j \eta\left(\frac{\log T}{2\pi}(\gamma_j - t)\right) dt \approx \sum_{T \leq \gamma \leq 2T} \frac{1}{T}\int_{-\infty}^{\infty} \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right) dt$$

$$= \frac{N(2T)-N(T)}{T} \frac{\int \eta}{\frac{\log T}{2\pi}} \sim \frac{T\frac{\log T}{2\pi}}{T} \frac{\int \eta}{\frac{\log T}{2\pi}} = \int \eta. \quad \square$$

Convention: (Fourier transform):

For $\varphi \in S(\mathbb{R}^k)$, $\hat{\varphi}(\xi) = \int_{\mathbb{R}} e(-x\cdot\xi)\varphi(x)\,d^kx$

for $\xi \in \mathbb{R}^k$

Then $\hat{\varphi} \in S(\mathbb{R}^k)$, and $\varphi(x) = \int_{\mathbb{R}^k} e(x\cdot\xi)\,\hat{\varphi}(\xi)\,d^k\xi$.

Thm (Rudnick-Sarnak): For $k \geq 2$, if

$$\operatorname{supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^k : |\xi_1| + \cdots + |\xi_k| < 2\}$$

then (GUE-corr-k) is true for φ.

We will prove (slightly weaker)

<u>Thm</u> (Hughes - Rudnick): For $k \geq 2$, if $\operatorname{supp} \hat{\eta} \subset \left(-\frac{2}{k}, \frac{2}{k}\right)$, then (GUE-moment-k) is true for η.

Intuition (uncertainty principle): The smaller $\operatorname{supp} \hat{\eta}$ is, the more smooth η must be.