Lecture 31

1. Zeta function and primes

Prime number thm: For $\pi(x) = \#\{\text{primes } p \le x\}$

$$\pi(x) \sim \frac{x}{\log x}$$

$\leftarrow$ random $n \approx x$ has prob. $\frac{1}{\log x}$ of being prime

Def: $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$

Def: $\psi(x) := \sum_{n \le x} \Lambda(n)$

PNT $\iff$ $\psi(x) \sim x$.

Zeta function connects to primes via:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \quad \text{for } \Re s > 1$$

← (encodes fund. thm. of arith.)

Implies $$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Thm (Riemann's explicit formula): For $x > 1$, with $x \notin \mathbb{N}$,

$$\psi(x) = x - \sum_{\substack{\sigma+i\gamma \\ \text{non-trivial} \\ \text{zeros}}} \frac{x^{\sigma+i\gamma}}{\sigma+i\gamma} - \log(2\pi) - \frac{1}{2}\log(1-x^{-2})$$

RH $\iff$ $\sigma = \frac{1}{2}$ always $\iff$ secondary term is $-x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}$

↖ roughly square root oscillations in error term

(Another heuristic viewpoint:

$$\Lambda(n) \approx 1 - n^{-\frac{1}{2}} \sum_{\gamma} n^{+i\gamma} \quad \text{on RH}$$
)

2. Guinand - Weil explicit formula

For the moment: Assume RH in what follows,
all nontrivial zeros of for $\frac{1}{2}+i\gamma$, $\gamma \in \mathbb{R}$

Thm (Guinand - Weil) Suppose $g \in C^{\infty}_{c}(\mathbb{R})$ (smooth and compactly supported). Then $\hat{g} \in S(\mathbb{R})$,

and

(over)

$$\sum_{\gamma} \hat{g}\left(\frac{\gamma}{2\pi}\right) - \int_{-\infty}^{\infty} \hat{g}\left(\frac{\xi}{2\pi}\right) \frac{\Omega(\xi)}{2\pi}\, d\xi$$

$$= -\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(g(\log n) + g(-\log n) \right) - \int_{0}^{\infty} \frac{1}{\sqrt{u}} \left(g(\log u) + g(-\log u) \right) du \right)$$

Where

$$\Omega(\xi) := \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + i\frac{\xi}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} - i\frac{\xi}{2}\right) \right) - \log \pi$$

<u>Remark</u>: Stirling's formula implies

$$\frac{\Omega(\xi)}{2\pi} = \frac{\log(|\xi|/2\pi)}{2\pi} + O\left(\frac{1}{|\xi|}\right)$$

for $|\xi| > 1$

Which is the density of γ near ξ.

So in explicit formula:

LHS = sum over zeros − smooth approx

RHS = sum over primes − smooth approx

Explicit formula = Fourier duality between <u>error terms</u> in zero and prime counting functions!

Idea behind proof: Functional eq. for Zeta

$$\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s)$$

Take log derivs $\rightsquigarrow$ relate $\dfrac{\zeta'(s)}{\zeta(s)}$ to $\dfrac{\zeta'(1-s)}{\zeta(1-s)}$ + gamma function terms

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n) = \sum\frac{\Lambda(n)}{\sqrt{n}}\int_{\mathbb{R}} e^{i2\pi y\log n}\hat{g}(y)\,dy$$

$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\underbrace{\sum\frac{\Lambda(n)}{n^{1/2+s}}}_{-\frac{\zeta'}{\zeta}\left(\frac{1}{2}+s\right)}\hat{g}\left(-\frac{s}{2\pi i}\right)ds$$

Shifting contour to left picks up "prime approx" at $s=1/2$. (over)

and picks up γ sum
at $\Re s = 0$.

Then use func. eq. and keep going left,
get gamma terms, and prime sum
at $-\log n$, minus
smooth approx. "$\square$"

(Details in Iwaniec-Kowalski's Ch. 5)

3. Linear statistics of Zeros

Want to examine: $\displaystyle\sum_{\gamma} \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right)$

If $\hat{g}\left(\frac{\xi}{2\pi}\right) = \eta\left(\frac{\log T}{2\pi}(\xi - t)\right)$

$$\Rightarrow g(x) = \int e^{i2\pi x\alpha}\,\hat{g}(\alpha)\,d\alpha =$$

$$= \int e^{i2\pi x\alpha}\,\eta\left(\frac{\log T}{2\pi}(2\pi\alpha - t)\right)d\alpha$$

$$\left(\begin{array}{l} \frac{\log T}{2\pi}(2\pi\alpha - t) = \beta \\ \alpha = \frac{\beta}{\log T} + \frac{t}{2\pi} \end{array}\right.$$

$$= \frac{e^{ixt}}{\log T}\int e^{i2\pi x\frac{\beta}{\log T}}\,\eta(\beta)\,d\beta$$

$$= \frac{e^{ixt}}{\log T}\,\hat{\eta}\left(-\frac{x}{\log T}\right) \leftarrow \text{to be applied}$$