Lecture 32

1. Linear statistics and primes

Last class: $\hat{g}\left(\frac{\xi}{2\pi}\right) = \eta\left(\frac{\log T}{2\pi}(\xi - t)\right) \iff g(x) = \frac{e^{ixt}}{\log T}\hat{\eta}\left(-\frac{x}{\log T}\right)$

Explicit formula $\Rightarrow$ (for $\hat{\eta} \in C_c^\infty(\mathbb{R})$) [On RH]

$$\left\{ \begin{aligned} &\sum_{\gamma} \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right) - \int_{\mathbb{R}} \eta\left(\frac{\log T}{2\pi}(\xi - t)\right) \frac{\Omega(\xi)}{2\pi} d\xi \\ &= -\frac{1}{\log T}\left[\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\left(n^{it}\hat{\eta}\left(-\frac{\log n}{\log T}\right) + n^{-it}\hat{\eta}\left(\frac{\log n}{\log T}\right)\right) \right. \\ &\qquad \left. - \int_0^\infty \frac{1}{\sqrt{u}}\left(u^{it}\hat{\eta}\left(-\frac{\log u}{\log T}\right) + u^{-it}\hat{\eta}\left(\frac{\log u}{\log T}\right)\right) du \right] \end{aligned} \right.$$

Claim I: For $\eta \in S(\mathbb{R})$, and $t \in [T, 2T]$

$$\int_{\mathbb{R}} \eta\left(\frac{\log T}{2\pi}(\xi - t)\right)\frac{\Omega(\xi)}{2\pi}\,d\xi = \int_{\mathbb{R}} \eta(y)\,dy + O\left(\frac{1}{\log T}\right)$$

Proof: Change of var. $y = \frac{\log T}{2\pi}(\xi - t)$, $d\xi = \frac{2\pi}{\log T}\,dy$

$$\text{LHS} = \int_{\mathbb{R}} \eta(y)\,\frac{\Omega\left(\frac{2\pi}{\log T}y + t\right)}{\log T}\,dy = \int_{\mathbb{R}} \eta(y)\,\frac{\log T + O(1+|y|)}{\log T}\,dy$$

$$= \int_{\mathbb{R}} \eta(y)\,dy + O\left(\frac{1}{\log T}\right) \quad \square$$

(Stirling's formula, and slow variation of log)

Claim II: For $\hat{\eta} \in C_c^\infty(\mathbb{R})$, and $t \in [T, 2T]$

$$\int_0^\infty \frac{u^{it}}{\sqrt{u}} \hat{\eta}\left(-\frac{\log u}{\log T}\right) du \, , \quad \int_0^\infty \frac{u^{-it}}{\sqrt{u}} \hat{\eta}\left(\frac{\log u}{\log T}\right) du = O_A\left(\frac{1}{T^A}\right)$$

for any $A > 0$.

Proof:

$$\int_0^\infty \frac{u^{it}}{\sqrt{u}} \hat{\eta}\left(\frac{-\log u}{\log T}\right) du \overset{u = e^x}{=} \int_{-\infty}^\infty e^{ixt} e^{x/2} \hat{\eta}\left(-\frac{x}{\log T}\right) dx$$

integration by parts ℓ times

$$= (-1)^\ell \int_{-\infty}^\infty \frac{e^{ixt}}{(it)^\ell} \frac{d^\ell}{dx^\ell}\left(e^{x/2} \hat{\eta}\left(\frac{-x}{\log T}\right)\right) dx$$

If $\hat{\eta}$ supported on $[-r,r]$, integral can be restricted to $|x| \leq r \log T$.

By estimating $\frac{d^\ell}{dx^\ell}(\sim)$ term, above is

$$= O_\ell\left(\frac{1}{t^\ell}\int_{-r\log T}^{r\log T} e^{x/2}\,dx\right)$$

$$= O_\ell\left(\frac{T^{r/2}}{T^\ell}\right)$$

$$= O_A\left(\frac{1}{T^A}\right) \leftarrow \text{as } \ell \text{ is arbitrary.} \qquad \square$$

for any $A > 0$

Lemma: For $\hat{\eta} \in C_c^\infty(\mathbb{R})$,

$$\sum_\gamma \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right) - \int_{\mathbb{R}} \eta(y)\,dy$$

$$= -\frac{1}{\log T}\left[\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2 - it}}\,\hat{\eta}\left(-\frac{\log n}{\log T}\right) + \frac{\Lambda(n)}{n^{1/2+it}}\,\hat{\eta}\left(\frac{\log n}{\log T}\right)\right]$$

$$+ O\left(\frac{1}{\log T}\right).$$

(from above claims)

2. Dirichlet polynomials

<u>Def</u>: A <u>Dirichlet polynomial</u> (in t) is a function $\sum_{n=1}^{N} a_n n^{-it}$

Philosophy: can get asymptotic estimates for

$$\frac{1}{T}\int_T^{2T} \Big|\sum_{n=1}^{N} a_n n^{-it}\Big|^{2k} dt$$ only when $N \ll \frac{T}{k}$

(for larger N, only get upper bounds, not always good)

Prop: $$\frac{1}{T}\int_T^{2T} n^{-it} m^{it}\, dt = \mathbb{1}_{n=m} + O\left(\frac{\sqrt{nm}}{T}\right)$$

for $n, m \in \mathbb{N}$

Proof: Let $\delta = \log(m/n)$, so above is

$$\frac{1}{T}\int_T^{2T} e^{i\delta t} = \frac{1}{T}\left(\frac{e^{i2\delta T} - e^{i\delta T}}{i\delta}\right)$$

If $n \neq m$, let $n > m$. Then

$$|\delta| = |\log(n/m)| > \left|\log\left(\frac{m+1}{m}\right)\right| = \left|\left(\frac{1}{m}\right) - \frac{1}{2}\left(\frac{1}{m}\right)^2 + \cdots\right|$$

$$\asymp \frac{1}{m}$$

So above is

$$= O\left(\frac{1}{|\delta| T}\right) = O\left(\frac{m}{T}\right)$$

$$= O\left(\frac{\sqrt{mn}}{T}\right). \quad \checkmark$$

If $n = m$, above is obviously

$$= 1. \quad \checkmark \quad \square$$

To come "Bohr identification" for p^{it}