Lecture 33

1. p^{it}, p prime

Warm up:

<u>Thm</u> (Kronecker - Weyl): If $\alpha_1, \ldots, \alpha_k$ are real numbers linearly independent over $\mathbb{Q}$, for $t \in [T, 2T]$ unif. dist.

$$\left(e^{i\alpha_1 t}, \ldots, e^{i\alpha_k t}\right) \xrightarrow{\text{dist}} (\omega_1, \ldots, \omega_k) \quad \text{as } T \to \infty$$

for $\omega_1, \ldots, \omega_k$ iid unif. dist. on $S^1 \subset \mathbb{C}$.

Proof: Moment method; Show for nonzero $(n_1, \dots, n_k) \in \mathbb{Z}^k$

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} (e^{i\alpha_1 t})^{n_1} \cdots (e^{i\alpha_k t})^{n_k} \, dt$$

$$= 0 \quad (= \mathbb{E}\, \omega_1^{n_1} \cdots \omega_k^{n_k})$$

True as $n_1 \alpha_1 + \cdots + n_k \alpha_k \neq 0$. $\square$

<u>Cor</u> (Bohr identification): For $t \in [T, 2T]$
unif. dist.

$$(2^{it}, 3^{it}, 5^{it}, \dots, p^{it}) \xrightarrow{dist} (X_2, X_3, X_5, \dots, X_p)$$

for $X_2, \dots, X_p$ iid unif dist. on $S^1 \subset \mathbb{C}$.

<u>Proof</u>: $\log 2, \log 3, \log 5, \dots, \log p$ lin. indep. over $\mathbb{Q}$. If not, there would be integers $e_1, \dots, e_p$ not all zero such that

$$e_2 \log 2 + e_3 \log 3 + \dots + e_p \log p = 0$$

$$\Rightarrow 2^{e_2} \cdots p^{e_p} = 1.$$

This would contradict Fund. Thm. of Arith. $\square$

2. A refinement on averaging

We have been considering averages

$$\frac{1}{T}\int_T^{2T} \sim\sim \, dt$$

Analysis often nicer if we consider

$$\frac{1}{T}\int_{\mathbb{R}} \sigma\left(\frac{t}{T}\right) \sim\sim \, dt$$

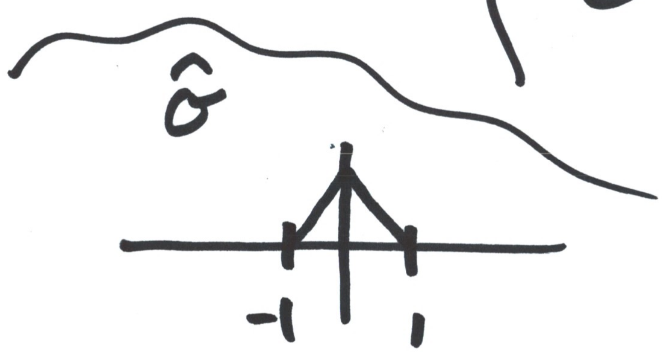
for a 'nice' function σ.

For $\sigma = \mathbb{1}_{[1,2]}$ this is the same.

We will consider σ such that $\hat{\sigma}$ is compactly supported.

<u>Ex</u>: If $\sigma(t) = \left(\frac{\sin \pi t}{\pi t}\right)^2$

then $\hat{\sigma}(\xi) = \begin{cases} 1 - |\xi| & \text{if } |\xi| < 1 \\ 0 & \text{otherwise} \end{cases}$



Prop: For $\sigma \in L^1(\mathbb{R})$ with $\int \sigma = 1$,
and $\hat{\sigma}$ compactly supported, if $\varepsilon > 0$
and $\min(m, n) \leq T^{1-\varepsilon}$,

$$\frac{1}{T} \int_{\mathbb{R}} \sigma\left(\frac{t}{T}\right) n^{-it} m^{it} \, dt = \mathbb{1}_{n=m}$$

for T suff. large (depending on σ, ε).

<u>Proof</u>:

$$LHS = \int_{\mathbb{R}} \sigma(\tau)\, e^{i\tau T \log(m/n)}\, d\tau$$

$$= \hat{\sigma}\left(\frac{T}{2\pi}\log(m/n)\right)$$

If $n \neq m$, with no loss of generality, suppose $n > m$. Then $|\log(m/n)| \gg \frac{1}{m} \geq \frac{1}{T^{1-\varepsilon}}$.

so $\hat{\sigma}\left(\frac{T}{2\pi}\cdot\log(m/n)\right) = 0$ for suff. large T.

If $n = m$, it's clear. $\square$

<u>Cor</u>: For σ as above,

$p_1, \dots, p_J, q_1, \dots, q_K$ primes with

$$\min(p_1 \cdots p_J, q_1 \cdots q_K) \le T^{1-\varepsilon},$$

then

$$\frac{1}{T}\int_{\mathbb{R}} \sigma\left(\frac{t}{T}\right) \prod_{j=1}^{J} p_j^{it} \prod_{k=1}^{K} q_k^{-it}\, dt = \mathbb{1}\left(\begin{array}{l}\text{there is a} \\ \text{matching} \\ p_1 = q_{\sigma(1)},\ p_2 = q_{\sigma(2)}, \dots \\ \text{for } \sigma \in S_J\end{array}\right)$$

for T suff. large.

Proof: $\text{LHS} = \mathbb{1}(p_1 \cdots p_J = q_1 \cdots q_K)$

$= \mathbb{1}(\text{matching})$

by Fund. Thm. of Arith., for suff. large T. $\square$

Note RHS is

$$\mathbb{E} \prod_{j=1}^{J} X_{p_j} \prod_{k=1}^{K} \overline{X}_{q_k}$$

for $X_2, X_3, \ldots$ iid, unif. dist. on $S' \subset \mathbb{C}$.

3. Prime sums

Lemma: If $f \in C_c^\infty(\mathbb{R})$

$$\lim_{T\to\infty} \frac{1}{\log T} \sum_p \frac{\log p}{p} f\left(\frac{\log p}{\log T}\right) = \int_0^\infty f(u)\, du$$

Proof: PNT $\Rightarrow \displaystyle\sum_{p \le x} \frac{\log p}{p} \sim \log x$

So $\displaystyle\sum_p \frac{\log p}{p} \mathbb{1}_{[0,\alpha]}\left(\frac{\log p}{\log T}\right) \sim \log(T^\alpha) = \log T \int \mathbb{1}_{[0,\alpha]}$

Now approximate

$$\left| f - \sum c_{\alpha_k} \mathbb{1}_{[0,\alpha_k]} \right| \leq \varepsilon \leftarrow \text{arb. small}. \quad \square$$

3. Mean square of Hughes - Rudnick Dirichlet poly

Lemma: For σ as before. Suppose $\eta : \mathbb{R} \to \mathbb{R}$, so $\hat{\eta}(-\xi) = \overline{\hat{\eta}(\xi)}$. If $\operatorname{supp} \hat{\eta} \subset [-\delta, \delta]$ with $\delta < 1$, then

$$\frac{1}{T} \int \sigma(t/T) \left(\frac{1}{\log T} \sum_p \frac{\log p}{p^{1/2 - it}} \hat{\eta}\left(-\frac{\log p}{\log T}\right) + \frac{\log p}{p^{1/2+it}} \hat{\eta}\left(\frac{\log p}{\log T}\right) \right)^2 dt$$

$$\sim \int_{-\delta}^{\delta} |u| \, |\hat{\eta}(u)|^2 \, du .$$

Proof: For suff. large T,

$$\text{LHS} = \frac{2}{(\log T)^2} \sum_{p \le T^\delta} \frac{(\log p)^2}{p} \left| \hat{\eta}\left(\frac{\log p}{\log T}\right) \right|^2$$

$$\sim 2 \int_0^\delta |u|\,|\hat{\eta}(u)|^2\,du$$

$$= \int_{-\delta}^{\delta} |u|\,|\hat{\eta}(u)|^2\,du. \qquad \square$$