Supplement to Lecture 34

More on the tauberian argument:

Have shown for fixed $\sigma \in L^1(\mathbb{R})$ with $\hat{\sigma}$ compactly supported, and fixed η, P as in Lecture 34,

(a) $$\lim_{T\to\infty} \int \frac{\sigma(t/T)}{T} P^k \, dt = c_k \left(\int \sigma\right) \left(\int_{-\delta}^{\delta} |u| \, |\hat{\eta}(u)|^2 \, du \right)^{k/2}.$$

(We've only shown this for $\int \sigma = 1$, but (a) follows by scaling.)

We want to show

(b) $$\lim_{T\to\infty} \int \frac{\mathbb{1}_{[1,2]}(t/T)}{T} P^k \, dt = c_k \left(\int_{-\delta}^{\delta} |u| \, |\hat{\eta}(u)|^2 \, du \right)^{k/2}.$$

Prove with two claims:

<u>Claim I</u>: For any $\varepsilon > 0$, there exists σ_1 with $\sigma_1 \in L^1(\mathbb{R})$ and $\hat{\sigma}_1$ compactly supported such that

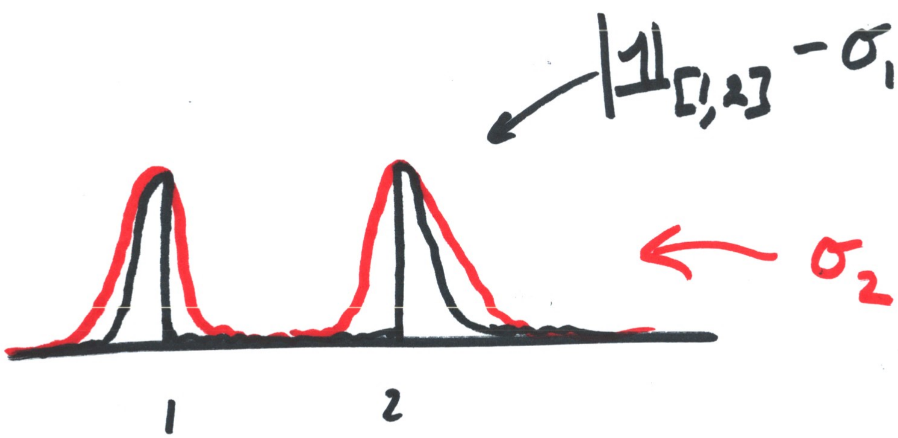
$$\| \mathbb{1}_{[1,2]} - \sigma_1 \|_{L^1(\mathbb{R})} < \varepsilon/2 .$$

(This follows by L^1 convergence of the Fourier transform for e.g. the Fejér Kernel, for those who have studied Fourier analysis, but it can be seen more elementarily by taking $\sigma_1(t)$ to be linear combinations of translations and dilations of the function $h(t) = \left(\frac{\sin \pi t}{\pi t}\right)^2$.)

← 3 translated/dilated copies of σ_1

Claim II: For σ_1 as above, there exists σ_2, a linear combination of translations and dilations of the function h, such that σ_2 is non-negative and

$$\left|\mathbb{1}_{[1,2]}(t) - \sigma_1(t)\right| \leq \sigma_2(t) \quad \text{and} \quad \|\sigma_2\|_{L^1(\mathbb{R})} < \varepsilon$$



These claims are clear from these pictures, (4)

and with them, by (a)

$$\lim_{T\to\infty} \int \frac{\sigma_2(t/T)}{T} |P|^k dt \le \lim_{T\to\infty} \left(\int \frac{\sigma_2(t/T)}{T} P^{2k} dt \right)^{\frac{1}{2}} = O_k(\sqrt{\varepsilon}).$$

Thus

$$\lim_{T\to\infty} \int \left(\frac{\mathbb{1}_{[1,2]}(t/T)}{T} - \frac{\sigma_1(t/T)}{T} \right) P^k dt = O_k(\sqrt{\varepsilon})$$

This gives

$$\lim_{T\to\infty} \int \frac{\mathbb{1}_{[1,2]}(t/T)}{T} P^k dt = c_k \left(\int \sigma_1 \right) \left(\int_{-\delta}^{\delta} |u| |\hat{\eta}(u)|^2 du \right)^{k/2} + O_k(\sqrt{\varepsilon}).$$

But by Claim I again, $\int \sigma_1 = \int 1\!\!1_{[1,2]} + O(\varepsilon) = 1 + O(\varepsilon)$.

Hence

$$\lim_{T\to\infty} \int \frac{1\!\!1_{[1,2]}(t/T)}{T} P^k \, dt = C_k \left(\int_{-\delta}^{\delta} |u| |\hat{\eta}(u)|^2 \, du \right)^{k/2} + O_k(\varepsilon + \sqrt{\varepsilon}).$$

As ε is arbitrary, (b) follows.