Lecture 34

1. Higher moments of the Hughes-Rudnick Dirichlet polynomial

Def: $c_k = \begin{cases} (k-1)(k-3)\cdots 3\cdot 1 & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$ $k \geq 0$

Note: In a set with k elements, c_k is the number of ways to match up $\frac{k}{2}$ elements with $\frac{k}{2}$ elements.

e.g. $\{1,2,3,4\}$: matchings $1 \leftrightarrow 2, 3 \leftrightarrow 4$; $1 \leftrightarrow 3, 2 \leftrightarrow 4$; $1 \leftrightarrow 4, 2 \leftrightarrow 3$.

Also, for $G \sim N_{\mathbb{R}}(0,1)$, $\mathbb{E}G^k = \int_{\mathbb{R}} x^k e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = c_k$ (Ex in integration by parts)

Prop: For $\sigma \in L^1(\mathbb{R})$ with $\int \sigma = 1$ and $\hat{\sigma}$ of compact support, suppose $\eta : \mathbb{R} \to \mathbb{R}$ so $\hat{\eta}(-\xi) = \overline{\hat{\eta}(\xi)}$.

If $\operatorname{supp} \hat{\eta} \subset [-\delta, \delta]$ with $\delta < 2/k$,

$$(*) \quad \lim_{T \to \infty} \int \frac{\sigma(t/T)}{T} \underbrace{\left(\frac{1}{\log T} \sum_{\varepsilon = \pm 1} \sum_{p} \frac{\log p}{p^{1/2}} p^{i\varepsilon t} \hat{\eta}\left(-\varepsilon \frac{\log p}{\log T}\right) \right)^k}_{:= Q_1^k} dt$$

$$= C_k \left(\int_{-\delta}^{\delta} |u| |\hat{\eta}(u)|^2 du \right)^{k/2}$$

Proof:

$$Q_1^k = \frac{1}{(\log T)^k} \sum_{\varepsilon_1, \dots, \varepsilon_k = \pm 1} \sum_{p_1, \dots, p_k} \frac{\log p_1 \cdots \log p_k}{\sqrt{p_1 \cdots p_k}} p_1^{i\varepsilon_1 t} \cdots p_k^{i\varepsilon_k t} \cdot \hat{\eta}\left(-\varepsilon_1 \frac{\log p_1}{\log T}\right) \cdots \hat{\eta}\left(-\varepsilon_k \frac{\log p_k}{\log T}\right)$$

For suff. large T, a term

$$\int \frac{\sigma(t/T)}{T} p_1^{i\varepsilon_1 t} \cdots p_k^{i\varepsilon_k t}\, dt$$

survives iff there are matchings for all primes:

$p_i = p_j$ with $\varepsilon_i = -\varepsilon_j$.

(Using that $p_j \leq T^\delta$ for all j.)

Thus

$$\text{LHS of } (*) = \frac{2^{k/2} c_k}{(\log T)^k} \left(\sum_p \frac{(\log p)^2}{p} \left| \hat{\eta}\left(\frac{\log p}{\log T}\right)\right|^2 \right)^{k/2}$$

$$\sim c_k \left(\int_{-\delta}^{\delta} |u| \, |\hat{\eta}(u)|^2 \, du \right)^{k/2} \qquad \square$$

Now $\Lambda(p) = \log p$, but must also consider $\Lambda(p^\ell) = \log p$ for $\ell \geq 2$.

<u>Prop</u>: For σ and η as above, if

$$Q_\ell = \frac{1}{\log T} \sum_{\varepsilon = \pm 1} \sum_{p} \frac{\log p}{p^{\ell/2}} \, p^{i\varepsilon \ell t} \, \hat{\eta}\left(-\varepsilon \frac{\log p^\ell}{\log T}\right)$$

then for $\ell \geq 2$

$$\int \frac{\sigma(t/T)}{T} (Q_\ell)^k \, dt = O_k\left(\frac{1}{(2^{\ell/2} \log T)^k}\right)$$

as $T \to \infty$.

Proof: Same reasoning above, for suff. large (5)

T,

$$\int \frac{\sigma(t/T)}{T} (Q_\ell)^k \, dt = \frac{2^{k/2} c_k}{(\log T)^k} \Big(\sum_p \frac{(\log p)^2}{p^\ell} \Big| \hat{\eta}\Big(\frac{\log p^\ell}{\log T} \Big) \Big|^2 \Big)^{k/2}$$

convergent sum

$$= O_k \left(\frac{1}{(\log T)^k \, 2^{\ell k/2}} \right).$$

$\square$

Thm: For σ and η as above, if

$$P = \frac{1}{\log T} \sum_{\varepsilon = \pm 1} \sum_{n} \frac{\Lambda(n)}{n^{1/2}} n^{i\varepsilon t} \hat{\eta}\left(-\varepsilon \frac{\log n}{\log T}\right)$$

then

$$\lim_{T \to \infty} \int \frac{\sigma(t/T)}{T} P^k \, dt = c_k \left(\int_{-\delta}^{\delta} |u| \, |\hat{\eta}(u)|^2 \, du \right)^{k/2} + o(1).$$

<u>Proof</u>: Note

$$P = Q_1 + Q_2 + \cdots$$

Use notation $\|P\|_k = \left(\int \frac{\sigma(t/T)}{T} |P|^k dt\right)^{1/k}$.

For k even

$$\big| \|P\|_k - \|Q_1\|_k \big| \leq \|P - Q_1\|_k$$

$$\leq \|Q_2\|_k + \|Q_3\|_k + \cdots$$

$$= O_k\left(\frac{1}{\log T}\left(\frac{1}{2} + \frac{1}{2^{3/2}} + \cdots\right)\right)$$

$$\to 0$$

Implies claim for k even.

Also implies

$$\|P\|_k, \|Q_1\|_k = O_k(1)$$

for all k, by Hölder.

For k odd:

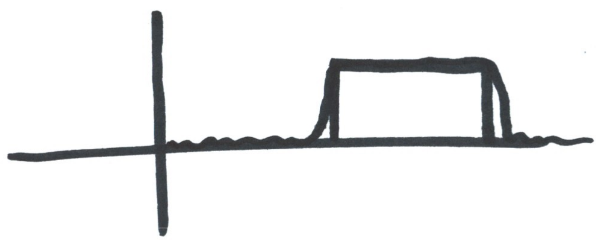
$$\int \frac{\sigma(t/T)}{T}(P^k - Q_1^k)\,dt = \int \frac{\sigma(t/T)}{T}(P-Q_1)(P^{k-1} + \cdots + Q_1^{k-1})\,dt$$

$$= O_k(\|P-Q_1\|_2) \to 0$$

Implies claim for k odd. $\square$

How to go from $\int \frac{\sigma(t/T)}{T} \sim$ to $\frac{1}{T}\int_T^{2T} \sim$

Idea (tauberian argument): Approximate $\mathbb{1}_{[1,2]}$ by σ with $\hat{\sigma}$ of compact support



<u>Prop</u>: For P as in the Thm,

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} P^k \, dt = C_k \left(\int_{-\delta}^{\delta} |x| \, |\hat{\eta}(x)|^2 dx \right)^{k/2}.$$