Lecture 35

1. Moments of linear statistics of zeros

We've shown: For $t \in [T, 2T]$, $\hat{\eta} \in C_c^\infty(\mathbb{R})$

$\eta : \mathbb{R} \to \mathbb{R}$

(Lec. 32)
$$\sum_{\gamma} \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right) - \int \eta(y)\,dy$$

$$= \underbrace{-\frac{1}{\log T} \sum_{\varepsilon = \pm 1} \sum_{n \geq 1} \frac{\Lambda(n)}{n^{1/2}} n^{i\varepsilon t} \hat{\eta}\left(-\varepsilon \frac{\log n}{\log T}\right)}_{P} + O\left(\frac{1}{\log T}\right)$$

Where if $\operatorname{supp} \hat{\eta} \subset \left(-\frac{2}{k}, \frac{2}{k}\right)$

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} P^k \, dt = c_k \left(\int_{-2/k}^{2/k} |u| \, |\hat{\eta}(u)|^2 \, du \right)^{k/2}$$

<u>Thm</u> (Hughes - Rudnick) (Assume RH) For η as above if $\ell \le k$,

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \left(\sum_{\gamma} \eta\left(\frac{\log T}{2\pi}(\gamma - t)\right) - \int \eta \right)^{\ell} dt = c_\ell \left(\int |u| \, |\hat{\eta}(u)|^2 \, du \right)^{\ell/2}$$

("Mock-gaussian behavior")

2. Comparison to sine-kernel process

Thm: For $\{\xi_k\}$ configs of the sine-kernel process and $\operatorname{supp}\hat{\eta} \subset \left(-\frac{2}{k}, \frac{2}{k}\right)$, for $\ell \le k$

$$\mathbb{E}\left(\sum_k \eta(\xi_k) - \int \eta\right)^{\ell} = c_\ell \left(\int |u| |\hat{\eta}(u)|^2 du\right)^{\ell/2}$$

Idea of proof: Use $\{N\vartheta_k\} \to \{\xi_k\}$ for ϑ_k e-angles $\in [-\frac{1}{2}, \frac{1}{2})$ of $g \in \mathcal{U}(N)$.

For

$$f : [-\tfrac{1}{2}, \tfrac{1}{2}) \to \mathbb{R} \quad \text{defined by} \quad f(x) = \eta(Nx)$$

$$\eta(Nx) = f(x) = \sum_{m \in \mathbb{Z}} e(mx)\, \hat{f}_m = \sum_{m} e(mx) \int_{-1/2}^{1/2} \eta(Ny)\, e(-my)\, dy$$

$$\approx \sum_{m} e(mx) \frac{1}{N} \int_{-\infty}^{\infty} \eta(y)\, e\left(-\frac{m}{N} y\right) dy$$

$$= \frac{1}{N} \sum_{m} e(mx)\, \hat{\eta}\left(\frac{m}{N}\right) = \frac{1}{N} \int \eta + \frac{1}{N} \sum_{m \neq 0} e(mx)\, \hat{\eta}\left(\frac{m}{N}\right)$$

Thus

$$\left[\begin{aligned} \sum_{k=1}^{N} \eta(N\vartheta_k) - \int \eta &\approx \frac{1}{N} \sum_{m \neq 0} \mathrm{Tr}(g^m)\, \hat{\eta}\left(\frac{m}{N}\right) \\ &= \frac{2}{N} \Re \sum_{m \geq 1} \mathrm{Tr}(g^m)\, \hat{\eta}\left(\frac{m}{N}\right) \end{aligned} \right]$$

Recall Diaconis–Shahshahani (slightly rewritten):

for $\min\left(\sum m_j, \sum n_k\right) \leq N$

$$\int_{\mathcal{U}(N)} \prod_{j=1}^{J} \mathrm{Tr}(g^{m_j}) \prod_{k=1}^{K} \overline{\mathrm{Tr}(g^{n_k})}\, dg = \mathbb{E} \prod_{j=1}^{J} \sqrt{m_j}\, G_{m_j} \prod_{k=1}^{K} \overline{\sqrt{n_k}\, G_{n_k}}$$

for $G_1, G_2, \ldots$ iid, $G_i \sim N_{\mathbb{C}}(0,1)$.

Hence for $\operatorname{supp} \hat{\eta} \subset \left(-\frac{2}{k}, \frac{2}{k}\right)$, for $\ell \leq k$

$$\int_{\mathcal{U}(N)} \left(\sum_{k=1}^{N} \eta(N\vartheta_k) - \int \eta \right)^{\ell} dg \approx \mathbb{E}\left(\frac{2}{N} \mathcal{R} \sum_{m \geq 1} \sqrt{m}\, G_m\, \hat{\eta}\left(\frac{m}{N}\right) \right)^{\ell}$$

sum of powers in traces are small enough that D-S applies.

Now

$$\frac{2}{N}\Re\sum_{m\geq 1}\sqrt{m}\,G_m\,\hat{\eta}\left(\frac{m}{N}\right) \overset{dist}{\sim} 2\Re N_{\mathbb{C}}\left(0, \sum_{m\geq 1}\frac{m}{N^2}\left|\hat{\eta}\left(\frac{m}{N}\right)\right|^2\right)$$

$$\left(\begin{array}{l}\text{Recall}\\ \Re N_{\mathbb{C}}(0,\sigma^2) \sim N_{\mathbb{R}}\left(0,\frac{\sigma^2}{2}\right)\end{array}\right) \quad \overset{dist}{\sim} N_{\mathbb{R}}\left(0, 2\sum_{m\geq 1}\frac{m}{N^2}\left|\hat{\eta}\left(\frac{m}{N}\right)\right|^2\right)$$

Note $$2\sum_{m\geq 1}\frac{m}{N^2}\left|\hat{\eta}\left(\frac{m}{N}\right)\right|^2 \longrightarrow 2\int_0^\infty u\,|\hat{\eta}(u)|^2\,du = \int_{-\infty}^{\infty}|u|\,|\hat{\eta}(u)|^2\,du$$

as $N \to \infty$ (Riemann sum).

Thus

$$\lim_{N\to\infty} \int_{\mathcal{U}(N)} \left(\sum_{k=1}^{N} \eta(N\vartheta_k) - \int \eta \right)^{\ell} dg$$

$$= c_\ell \left(\int |u| \, |\hat{\eta}(u)|^2 \, du \right)^{\ell/2} .$$

real gaussian moments. $\square$

This proves the microscopic

Zeta Zero $\longleftrightarrow$ RMT correspondence

against sufficiently bandlimited functions.

Q: What if $\operatorname{supp} \hat{\eta} \not\subset \left(-\frac{2}{\ell}, \frac{2}{\ell}\right)$?

A: Pattern becomes much more complex than gaussian moments!

Even variance changes:

$$\int_{U(N)} |\mathrm{Tr}(g^m)|^2 \, dg = \min(|m|, N) \quad \text{for } m \neq 0$$

Leads to a variance for sine-kernel process:

$$\int_{-\infty}^{\infty} \min(|x|, 1) \, |\hat{\eta}(x)|^2 \, dx.$$

A relevant result in RMT:

<u>Thm</u> (Rains): For $g \in \mathcal{U}(N)$ for $m > N$, $\omega_1^m, \ldots, \omega_N^m$ will be iid, unif. dist. on $S^1 \subset \mathbb{C}$.