Lecture 4

1. A construction of Haar measure on $O(N)$

Idea: Fill up an $N \times N$ random matrix X with iid real Gaussian entries: $X_{ij} \sim N_{\mathbb{R}}(0,1)$. Perform Gram-Schmidt on the columns of X. Resulting matrix is a Haar distributed random orthogonal matrix.

Recall Gram-Schmidt: Given N linearly independent vectors $v_1, v_2, .., v_N \in \mathbb{R}^N$, form N orthonormal vectors $e_1, e_2, .., e_N \in \mathbb{R}^N$ with

$$\text{span}\{e_1, .., e_\ell\} = \text{span}\{v_1, .., v_\ell\} \text{ for all } \ell$$

by

$$u_1 = v_1, \quad e_1 = u_1/\|u_1\|$$

$$u_2 = v_2 - \langle v_2, e_1\rangle e_1, \quad e_2 = u_2/\|u_2\|$$

$$u_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2,$$

$$e_3 = u_3/\|u_3\|$$

etc.

For Gram-Schmidt on vectors $v_1, \ldots, v_N$, ③

let

$$T(v_1, \ldots, v_N) := (e_1, \ldots, e_N)$$

be the output vectors of Gram-Schmidt

For a matrix $M = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_N \\ | & & | \end{pmatrix} \in GL_N(\mathbb{R})$

$$T(M) := \begin{pmatrix} | & & | \\ e_1 & \cdots & e_N \\ | & & | \end{pmatrix}$$

be the output of doing Gram-Schmidt on column vectors.

Note $T(M) \in O(N)$ always.

<u>Claim 1</u>: If $h \in O(N)$, $M \in GL_N(\mathbb{R})$

$$T(hM) = hT(M)$$

(for h, M any metrics).

That is: Gram-Schmidt preserves rotations/flips of orientation.

Can be proved formally by induction, but evident geometrically!

⑤

Claim 2: If $X = (X_{ij})$ is an $N \times N$ random matrix with iid entries $X_{ij} \sim N_{\mathbb{R}}(0,1)$ $\forall ij$ (the real Ginibre ensemble), for $h \in O(N)$ deterministic the random matrix hX has the same dist. as X.

What does it mean? For any continuous $\varphi : \mathrm{Mat}_{N\times N}(\mathbb{R}) \to \mathbb{C}$,

$$\mathbb{E}\varphi(hX) = \mathbb{E}\varphi(X)$$

Why?

$$\mathbb{E}\varphi(hX) = \frac{1}{(2\pi)^{N^2/2}} \int_{\mathbb{R}^{N^2}} \varphi(h(x_{ij})) \cdot \exp\left(-\frac{(x_{11}^2 + \cdots + x_{NN}^2)}{2}\right) d^{N^2}x$$

all N^2 entries

$$(x_{ij}) = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_N \\ | & & | \end{pmatrix}$$

$$= \frac{1}{(2\pi)^{N^2/2}} \int_{\mathbb{R}^{N^2}} \varphi\begin{pmatrix} | & & | \\ hx_1 & \cdots & hx_N \\ | & & | \end{pmatrix} \cdot \exp\left(-\frac{\|x_1\|^2 + \cdots + \|x_N\|^2}{2}\right) d^{N^2}x$$

as h is an isometry $\rightarrow$

$$= \frac{1}{(2\pi)^{N^2/2}} \int_{\mathbb{R}^{N^2}} \varphi\begin{pmatrix} | & & | \\ y_1 & \cdots & y_N \\ | & & | \end{pmatrix} \cdot \exp\left(-\frac{\|y_1\|^2 + \cdots + \|y_N\|^2}{2}\right) d^{N^2}y$$

$$= \mathbb{E}\varphi(X) \qquad \square$$

One can reformulate this: for any Borel set A

$$\mathbb{P}(X \in A) = \mathbb{P}(hX \in A).$$

<u>Thm</u>: For X a $N \times N$ random matrix as above ($X_{ij} \sim N_{\mathbb{R}}(0,1)$ $\forall i j$), the random matrix $g = T(X)$ obtained by Gram-Schmidt on the columns is a left-Haar dist. random matrix of $O(N)$.

(Note: a.s. $X \in GL_N(\mathbb{R})$, so a.s. g is well-defined.)

<u>Proof</u>: Plain $g \in O(N)$, so need only show for Borel A, and any deterministic $h \in O(N)$,

$$\mathbb{P}(hg \in A) = \mathbb{P}(g \in A)$$

$$\Updownarrow$$

$$\mathbb{P}(T(hX) \in A) = \mathbb{P}(hT(X) \in A) = \mathbb{P}(T(X) \in A)$$

claim 1

$$\Updownarrow$$

$$\mathbb{P}(hX \in T^{-1}(A)) = \mathbb{P}(X \in T^{-1}(A))$$

claim 2

$\square$

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Note: Gram-Schmidt on the rows of X produces a right-Haar dist. random matrix. By uniqueness proof last class, left-Haar = right-Haar, and Haar measure is unique.

2. Haar measure on $U(N)$ and $Sp(2N)$

- Construction for $U(N)$ is the same, but take X complex:

$$X_{ij} \sim N_{\mathbb{C}}(0,1) = N_{\mathbb{R}}(0,\tfrac{1}{2}) + iN_{\mathbb{R}}(0,\tfrac{1}{2}) \quad \forall ij$$

- Construction for $Sp(2N)$ is also the same, but more subtle: take X random with quaternion entries

$$X_{ij} \sim N_{\mathbb{H}}(0,1) = N_{\mathbb{R}}(0,\tfrac{1}{2}) + \dots + kN_{\mathbb{R}}(0,\tfrac{1}{2})$$

Must prove Gram-Schmidt makes sense over $\mathbb{H}^N$. Left to the interested!