Lecture 6

1. Dist. of matrix entries of $U(N)$, $Sp(2N)$

<u>Prop</u>: For random $g \in \mathcal{U}(N)$, the (complex-valued) $\langle u, gv \rangle$ has an ident. dist. for all unit vectors $u, v \in \mathbb{C}^N$. Hence g_{ij} has ident. dist. $\forall ij$.

Analogous statement for $g \in Sp(2N)$.

Proof: Unitary case: Model proof on $O(N)$. ②

To show

$$\langle u, gv\rangle \sim \langle u', gv'\rangle$$

u, v, u', v' unit vectors

need show

$$\exists h, k \in \mathcal{U}(N) \text{ s.t. } u' = hu, \; v' = kv.$$

(that is $\mathcal{U}(N)$ is transitive on $\mathbb{C}^N$.)

Let

h_a have first column u', so $h_a e_1 = u'$ ← standard basis vector

h_b " " u, so $h_b e_1 = u$

$h_a, h_b \in \mathcal{U}(N)$ | Thus $u' = h_a h_b^{-1} u$ (same for k)

Symplectic case: Again just need $Sp(2N)$ is transitive on unit ball of $\mathbb{C}^{2N}$.

Enough for $u' \in \mathbb{C}^{2N}$ a unit vector to find $h \in Sp(2N)$ with

$$u' = he_1 \iff h = \begin{pmatrix} | & \\ u' & \cdots \\ | & \end{pmatrix}$$

But

$$h \in Sp(2N) \iff h \in \mathcal{U}(2N),\ h = \begin{pmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{pmatrix}$$

$A, B, C, D \in Mat_{N\times N}(\mathbb{C})$

(Lectures 2+3)

So $\exists h = \begin{pmatrix} | & \\ u' & \cdots \\ | & \end{pmatrix} \in Sp(2N)$ by choosing matching blocks. $\square$

<u>Thm</u>: For random $g \in \mathcal{U}(N)$

$$g_{11} \sim Y_1 + i Y_2$$

for $Y = (Y_1, \ldots, Y_{2N}) \in \mathbb{R}^{2N}$ u.d. on S^{2N-1}.

<u>Thm</u>: For random $g \in Sp(2N)$

$$g_{11} \sim Y_1 + i Y_2$$

for $Y = (Y_1, \ldots, Y_{4N}) \in \mathbb{R}^{4N}$ u.d. on S^{4N-1}.

Proof: Gram-Schmidt process for $\mathcal{U}(N)$ gives first column of random $g \in \mathcal{U}(N)$

$$g = \begin{pmatrix} Y_1 + iY_2 & \\ Y_3 + iY_4 & \cdots \\ \vdots & \\ Y_{2N-1} + iY_{2N} & \end{pmatrix}$$

with Y ud on $Y_1^2 + Y_2^2 + \cdots + Y_{2N-1}^2 + Y_{2N}^2 = 1$ ✓

Gram-Schmidt for $Sp(2N)$ realized via $\mathbb{H}$ gives

$$\begin{pmatrix} Y_1 + iY_2 + jY_3 + kY_4 & \\ \vdots & \cdots \\ Y_{4N-3} + \cdots + kY_{4N} & \end{pmatrix} \in Mat_{N \times N}(\mathbb{H})$$

with Y ud on S^{4N-1}

$$\Longleftrightarrow \left(\begin{array}{c|c} Y_1 + i Y_2 & \\ \vdots \quad \cdots & \cdots \\ Y_{4N-3} + i Y_{4N-2} & \\ \hline -Y_3 + i Y_4 & \\ \vdots \quad \cdots & \cdots \\ -Y_{4N-1} + i Y_{4N} & \end{array}\right) \in Mat_{2N\times 2N}(\mathbb{C})$$

$\square$

<u>Note</u> (Mult. coord. on S^{M-1}) For $Y \in \mathbb{R}^M$ ud on S^{M-1} the same proof as last time shows for , <u>k fixed</u>,

$$\left(\sqrt{M}\, Y_1, \ldots, \sqrt{M}\, Y_k\right) \xrightarrow{dist.} \left(Z_1, \ldots, Z_k\right)$$

for $Z_1, \ldots, Z_k \sim N_{\mathbb{R}}(0,1)$ iid.

(Diaconis - Freedman showed this is true uniformly as long as $k = o(n)$.)

<u>Note</u> (complex gaussians): Recall

$$W \sim N_{\mathbb{R}}(0, \sigma^2) \iff \mathbb{P}(W \in A) = \int_A \frac{e^{-t^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\, dt$$

Have

$$Z \sim N_{\mathbb{C}}(0, \sigma^2) \iff \mathbb{P}(Z \in E) = \int_E \frac{e^{-|z|^2/\sigma^2}}{\sigma\pi}\, dx\,dy$$

Equivalently

$$Z \sim \underset{\nwarrow}{N_{\mathbb{R}}(0, \tfrac{\sigma^2}{2})} + i\, \underset{\nearrow}{N_{\mathbb{R}}(0, \tfrac{\sigma^2}{2})}, \quad \text{so } \mathbb{E}|Z|^2 = \sigma^2.$$

independent

Cor: For $g \in \mathcal{U}(N)$ as $N \to \infty$

$$\sqrt{N}\, g_{11} \xrightarrow{\text{dist.}} N_{\mathbb{C}}(0,1) \qquad \left(\begin{array}{c}\text{so} \\ g_{ij} \approx \dfrac{N_{\mathbb{C}}(0,1)}{\sqrt{N}}\end{array}\right)$$

For $g \in Sp(2N)$ as $N \to \infty$

$$\sqrt{2N}\, g_{11} \xrightarrow{\text{dist.}} N_{\mathbb{C}}(0,1) \qquad \left(\begin{array}{c}\text{so} \\ g_{ij} \approx \dfrac{N_{\mathbb{C}}(0,1)}{\sqrt{2N}}\end{array}\right)$$

In fact, for any fixed k, in all these cases $(O(N), \mathcal{U}(N), Sp(2N))$, $g_{11}, g_{21}, \ldots, g_{k1}$ tend to iid gaussians.

HW asks you to show g_{11}, g_{22} tend to iid gaussian in $O(N)$.

Proof will work not only for g_{11}, g_{22}, but for any $g_{11}, \ldots, g_{kk}$; $k \times k$ minor tends to $\frac{1}{\sqrt{N}}$ ($k \times k$ gaussian matrix) for any of $O(N)$, $U(N)$, $Sp(2N)$.

Meckes Ch. 2 explores how true this is if k grows with n.

Heuristically, what does this say about $\operatorname{Tr} g$, for $g \in O(N)$?

Expect

$$\operatorname{Tr} g = g_{11} + \cdots + g_{NN} \approx \frac{N_{\mathbb{R}}(0,1)}{\sqrt{N}} + \cdots + \frac{N_{\mathbb{R}}(0,1)}{\sqrt{N}}$$

$$\approx N_{\mathbb{R}}(0,1)$$

Actually true!

(A proof based on matrix entries due to Diaconis and d'Aristotile–Newman; we will discuss later from perspective of eigenvalues)