Lecture 7

1. Some important subgroups

Def: $SO(N) = \{g \in O(N) : \det(g) = 1\}$ ← "special orthogonal"

$SU(N) = \{g \in U(N) : \det(g) = 1\}$ ← "special unitary"

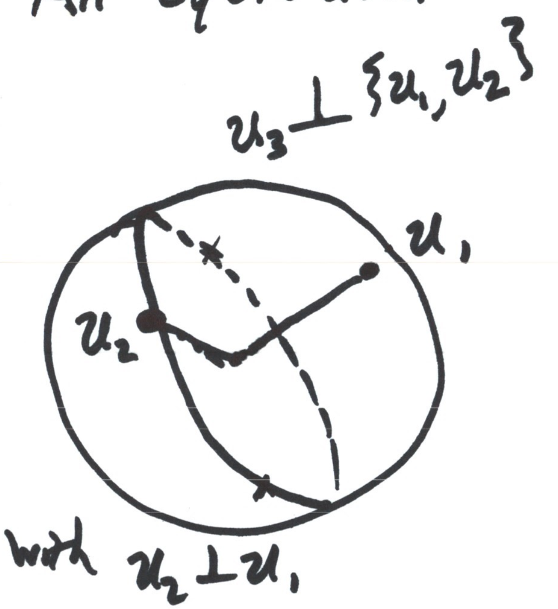
- Note for $g \in O(N)$, $\det(g) = \pm 1$, for $g \in U(N)$, $\det(g) = e^{i\theta}$
- Subgroups as $\det(g_1)\det(g_2) = \det(g_1 g_2)$
- No "special symplectic group" – we show $g \in Sp(2N) \Rightarrow \det(g) = 1$ later

Remark: $SO(N)$ is the collection of orientation-preserving isometries of $\mathbb{R}^N$ = group of rotations.

2. More on Haar measure

Recall Haar measure on $O(N)$: perform Gram-Schmidt on gaussian column vectors

An equivalent construction:



- choose a unit vector u_1 uniformly from S^{N-1} embedded in N-dim space
- choose a unit vector u_2 uniformly from unit sphere $\perp u_1$, embedded in a $N-1$ dim space
- choose unit vector u_3 uniformly from unit sphere $\perp \{u_1, u_2\}$ embedded in $N-2$ dim space

⋮

- choose u_N uniformly from the two antipodal points that remain

Then

$$g = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_N \\ | & | & & | \end{pmatrix} \text{ is Haar dist. on } O(N)$$

For Haar measure on $SO(N)$, do the same, but choose u_N so $\det\begin{pmatrix} | & & | \\ u_1 & \cdots & u_N \\ | & & | \end{pmatrix} = 1$.

Same idea for $U(N)$: choose $u_1, u_2, .., u_N$ from orth. subspaces of unit sphere of $\mathbb{C}^N$.

Can choose $u_N = e^{i\theta} u$, with θ chosen freely.

For Haar measure on $SU(N)$, choose θ so that

$$\det\begin{pmatrix} | & & | \\ u_1 & \cdots & u_N \\ | & & | \end{pmatrix} = 1.$$

3. A more careful look at SU(2) and SO(3)

$$SU(2) = \left\{ \begin{pmatrix} a & \vdots \\ b & \vdots \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2+|b|^2=1 \right\} \leftarrow \text{with second column uniquely determined}$$

$$= \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2+|b|^2=1 \right\}$$

$$= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2+|\beta|^2=1 \right\} \leftarrow \text{a more traditional way to write it.}$$

Topologically: $SU(2) \overset{\text{homeo}}{\cong} S^3$ (embedded in $\mathbb{R}^4$)

Observation: $SU(2)$ is simply connected.

$SO(3)$ = rotations in $\mathbb{R}^3$

Claim: topologically $SO(3) \overset{\text{homeo}}{\cong} S^3/\{-v \sim v\}$

(sphere with antipodal points identified

)

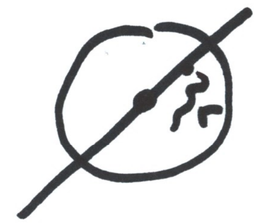
Why? For $D^3 \subseteq \mathbb{R}^3$ the unit ball,

first show $SO(3) \cong D^3/\{-v \sim v : v \in \underset{\partial D^3}{\underset{\parallel}{S^2}}\}$

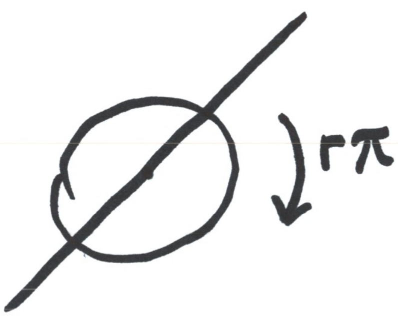
For, $p \in D^3$ corresponds to a line ℓ

through origin

and a scalar $r \in [-1, 1]$



To map into $SO(3)$, map $p \in D^3$ to a rotation about line ℓ in $\mathbb{R}^3$, by $r\pi$ radians.



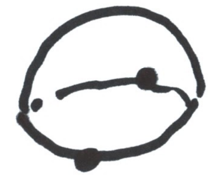
Corresponds one-to-one with rotations in $\mathbb{R}^3$, as long as $r=1$ and $r=-1$ identified.

$$\Rightarrow SO(3) \cong D^3 / \{\text{antipodal points on boundary identified}\}$$

For original claim:

sphere S^3 with antipodal equivalence

$\cong$ Closed hemisphere with antipodal equiv. on boundary

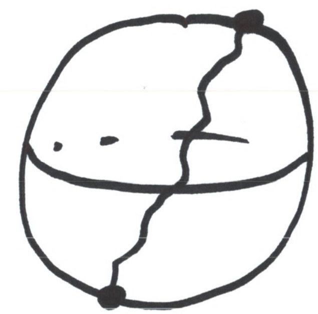


$\cong$ D^3 with antipodal equiv. on boundary

Observation:

SO(3) is not simply connected

(Consider path from point to antipode



- can't be retracted!)

But we have shown:

$$\text{Continuous } \psi : SU(2) \longrightarrow SO(3) \quad \text{two-to-one}$$

$$\left(\text{by } S^3 \to S^3/\{-v \sim v\}\right)$$

In fact we can find

$$\begin{array}{l}\text{Continuous}\\ \text{group}\\ \text{homomorphism}\end{array} \quad \varphi : \underset{\text{two-to-one}}{SU(2) \to SO(3)}$$

(see references) best done via quaternions

<u>Claim</u>: For $N \geq 3$, $SO(N)$ is not simply connected, but $SO(N)$ has a unique two-fold covering group $Spin(N)$ which is simply connected.

$$\left(\varphi : Spin(N) \to SO(N) \right)$$