Lecture 8

1. Classification of compact Lie groups (no proofs)

Def: A matrix Lie group is a closed subgroup of $GL(N, \mathbb{C})$ for some N.
(A Lie group is a top. group that is also a diff. manifold, with smooth group operations.)

Def: A group is simple if it has no non-trivial normal subgroup.

<u>Claim</u>: All simple simply-connected compact (matrix) Lie groups appear once in the following table

$SU(N)$	$N \geq 2$
$Sp(2N)$	$N \geq 2$
$Spin(N)$	$N \geq 7$
E_6	(subgroup of $GL(27, \mathbb{C})$)
E_7	($GL(56, \mathbb{C})$)
E_8	($GL(248, \mathbb{C})$)
F_4	($GL(26, \mathbb{C})$)
G_2	($GL(7, \mathbb{C})$)

(can remove "(matrix)" and still true.)

<u>Claim</u>: Every compact connected (matrix) Lie group G is of the form

$$G = K/H$$

where K is a finite product of $\mathbb{R}/\mathbb{Z}$ and the groups in the table above, and H is a discrete subgroup of the center of K.

(Moral: understanding Haar measure on any "big group" $\approx$ understanding Haar on $O(N)$, $U(N)$, $Sp(2N)$, or lots of independent products.)

2. One more look at Haar measure on SU(2)

Last class $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$

$$= \left\{ \underbrace{\begin{pmatrix} x_1 + i x_2 & x_3 + i x_4 \\ -x_3 + i x_4 & x_1 - i x_2 \end{pmatrix}}_{g(\theta,\varphi,\psi)} : x \in S^3 \right\}$$

Haar measure by choosing $x \in S^3$ uniformly

Set

$$x_1 = \cos\theta$$
$$x_2 = \sin\theta \cos\varphi$$
$$x_3 = \sin\theta \sin\varphi \cos\psi$$
$$x_4 = \sin\theta \sin\varphi \sin\psi$$

$\theta, \varphi \in [0, \pi]$

$\psi \in [0, 2\pi]$

Euler angles on S^3

Thm: For $F : SU(2) \to \mathbb{C}$

$$\int_{SU(2)} F(g)\,dg = \frac{1}{2\pi^2} \int_0^{\pi} d\theta \int_0^{\pi} d\varphi \int_0^{2\pi} d\psi \; F(g) \sin^2\theta \sin\varphi$$

(notation meaning $\mathbb{E} F(g)$, $g \in SU(2)$ Haar dist.)

Proof: Let σ be surface area on S^3.

Note $\sigma(S^3) = 2\pi^2$. Want

$$\frac{1}{2\pi^2} \int_{x \in S^3} F(x)\, d\sigma(x)$$

Let $y = rx$, $r \in [0,\infty)$

$$\int_{\mathbb{R}^4} F\left(\frac{y}{|y|}\right) e^{-\pi |y|^2} d^4 y = \int_0^{\infty} \left(\int_{x \in S^3} F(x)\, d\sigma(x) \right) r^3 e^{-\pi r^2} dr = \frac{1}{2\pi^2} \int_{x \in S^3} F(x)\, d\sigma(x)$$

But set

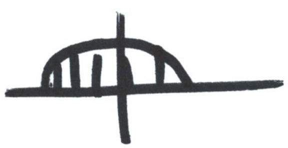
$$(y_1, y_2, y_3, y_4) = r\cdot x(\theta, \varphi, \psi) \Rightarrow d^4y = \underbrace{\left[\text{Jacobian}\right]}_{r^3 \sin^2\theta \sin\varphi}\cdot dr\, d\theta\, d\varphi\, d\psi$$

$$\Rightarrow \frac{1}{2\pi^2}\int_{x\in S^3} F(x)\,dx = \underbrace{\left(\int_0^\infty r^3 e^{-\pi r^2}\,dr\right)}_{1/2\pi^2}\iiint F(x)\,\sin^2\theta\,\sin\varphi\; d\theta\, d\varphi\, d\psi \qquad \square$$

Note g as before, $\mathrm{Tr}\, g = 2x_1$

<u>Cor</u>: $$\int_{SU(2)} f(\mathrm{Tr}\, g)\,dg = \frac{1}{2\pi^2}\int_0^\pi d\theta \int_0^\pi d\varphi \int_0^{2\pi} d\psi\, f(2\cos\theta)\,\sin^2\theta\,\sin\varphi$$

$$= \frac{1}{2\pi}\int_{-2}^{2} f(t)\sqrt{4-t^2}\,dt \;\swarrow$$ (Semicircular/ Sato-Tate distribution)



Note: if $g \in SU(2)$, g has e-values $\{e^{i\vartheta}, e^{-i\vartheta}\}$ for $\vartheta \in [0,\pi]$, and $\operatorname{Tr} g = 2\cos\vartheta$. Thus $\vartheta = \theta$, and ϑ has PDF $\frac{2}{\pi^2}\sin^2\theta$

Cor: If $\{\omega_1, \omega_2\}$ are the e-values of $g \in SU(2)$, and $f(\omega_1,\omega_2)$ is a symmetric function of the e-values

$$\mathbb{E} f(\omega_1,\omega_2) = \frac{1}{2}\int_{[0,2\pi)} f(e^{i\theta}, e^{-i\theta})\,\left|e^{i\theta}-e^{-i\theta}\right|^2 \frac{d\theta}{2\pi}$$

<u>Proof</u>:

$$RHS = \frac{1}{2}\int_0^{\pi} f(e^{i\theta}, e^{-i\theta})\left|e^{i\theta} - e^{-i\theta}\right|^2 \frac{d\theta}{\pi} = \int_0^{\pi} f(e^{i\theta}, e^{-i\theta}) \cdot \frac{2}{\pi}\sin^2\theta \, d\theta$$

A generalization:

<u>Thm</u> (Weyl integration formula): If $\omega_1, \omega_2, \ldots, \omega_N$ are e-values of $g \in \mathcal{U}(N)$ and $f(\omega_1, \ldots, \omega_N)$ is a symmetric function of e-values

$$\mathbb{E} f(\omega_1, \ldots, \omega_N) = \frac{1}{N!}\int_{[0,2\pi]^N} f(e^{i\theta_1}, \ldots, e^{i\theta_N}) \prod_{1 \le j < k \le N} \left|e^{i\theta_j} - e^{i\theta_k}\right|^2 \cdot \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}$$