Lecture 9

Goal in next few lectures: prove Weyl integration formula, giving joint density of eigenvalues

Preliminaries:

1. A norm on matrices

Def: For $X \in \mathrm{Mat}_{N\times N}(\mathbb{C})$, the Hilbert-Schmidt norm

$$\|X\|^2 = \|X\|_{HS}^2 := \sum_{i,j=1}^{N} |x_{ij}|^2$$

(unless otherwise noted, norms on matrices are HS norms)

Say: $X \to 0$ if $\|X\| \to 0$, $X = O(\varepsilon)$ if $\|X\| = O(\varepsilon)$.

Note: For $g \in \mathcal{U}(N)$, $\|gX\| = \|X\|$

Note: $\|XY\| \leq \|X\| \cdot \|Y\|$ $\leftarrow$ (Exercise in HW2)

2. Lie algebras

Recall: For $X \in Mat_{N \times N}(\mathbb{C})$,

$$\exp(X) = e^X := \sum_{k \geq 0} \frac{X^k}{k!}$$

converges (in HS norm)

For $\|Y - I\|$ small

$$\log(Y) := \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (Y - I)^k$$

inverse of e^X, proved via power series

- $e^X e^Y = e^{X+Y}$ if X and Y commute but $e^X e^Y \neq e^{X+Y}$ in general.

<u>Prop</u>: For a suff. small nbd. of $0 \in Mat_{N \times N}(\mathbb{C})$, exp continuously maps one-to-one onto a small nbd of I, with $\log(\exp X) = X$. (True also for $Mat_{N \times N}(\mathbb{C})$ replaced by $Mat_{N \times N}(\mathbb{R})$.)

<u>Prop</u>: For $\|X\|$ small,

$$e^X = I + X + O(\|X\|^2)$$

For $\|Y\|$ small, $\log(I+Y) = Y + O(\|Y\|^2)$

(Exercise in HW2)

<u>Cor</u>: $\lim_{k\to\infty} \left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)^k = e^{X+Y}$

<u>Proof</u>: $\log\left(e^{\frac{X}{k}} e^{Y/k}\right) = \log\left(I + \frac{X}{k} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right)\right)$

$$= \frac{X}{k} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right)$$

so

$$\left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)^k = \left(e^{\frac{X}{k} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right)}\right)^k = e^{X+Y+O\left(\frac{1}{k}\right)} \to e^{X+Y}.$$

$\square$

Def: For G a matrix Lie group (closed subgroup of $GL(N;\mathbb{C})$) the Lie algebra of G is

$$\mathfrak{g} = \mathrm{Lie}(G) := \{X \in Mat_{N\times N}(\mathbb{C}) : e^{tX} \in G \ \forall t \in \mathbb{R}\}$$

Prop: $\mathfrak{g}$ is a <u>real</u> vector space.

Proof: If $X, Y \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{R}$,

$\left(e^{\frac{t\alpha X}{k}} e^{\frac{t\beta Y}{k}}\right)^k \in G$ for all $t \in \mathbb{R}$, $k \in \mathbb{N}$

$\Rightarrow e^{t(\alpha X + \beta Y)} \in G \ \forall t$ by $k \to \infty$.

$\Rightarrow \alpha X + \beta Y \in \mathfrak{g}$. $\square$

<u>Ex</u>: $\mathfrak{u}(1) = \mathrm{Lie}(\mathcal{U}(1)) = \{ i\lambda : \lambda \in \mathbb{R} \}$

<u>Why</u>? $\mathcal{U}(1) = \{ e^{i\lambda} : \lambda \in \mathbb{R} \}$ and

$$\{ z \in \mathbb{C} : e^{tz} \in \mathcal{U}(1) \ \forall t \in \mathbb{R} \}$$
$$= \{ i\lambda : \lambda \in \mathbb{R} \}.$$

<u>Ex</u>: $\mathfrak{u}(N) = \mathrm{Lie}(\mathcal{U}(N)) = \{ X \in \mathrm{Mat}_{N\times N}(\mathbb{C}) : X^* + X = 0 \}$

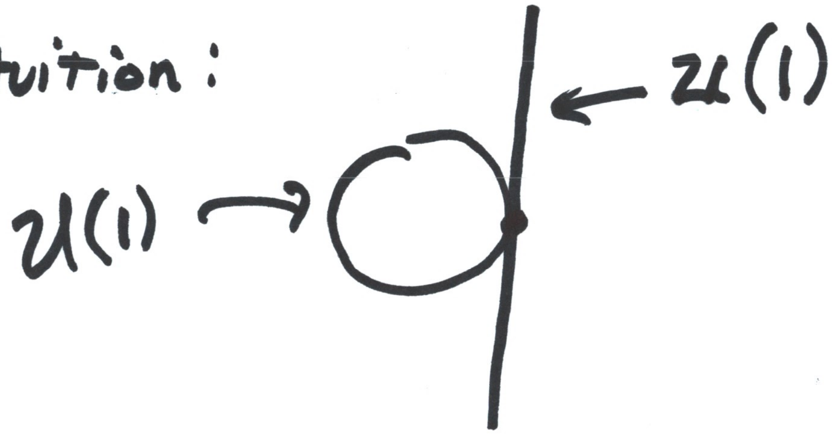
$$= \{ iH : H \text{ is } N\times N \text{ Hermitian} \}$$

Why? $\mathcal{U}(N) = \{ g^{-1} = g^* \}$

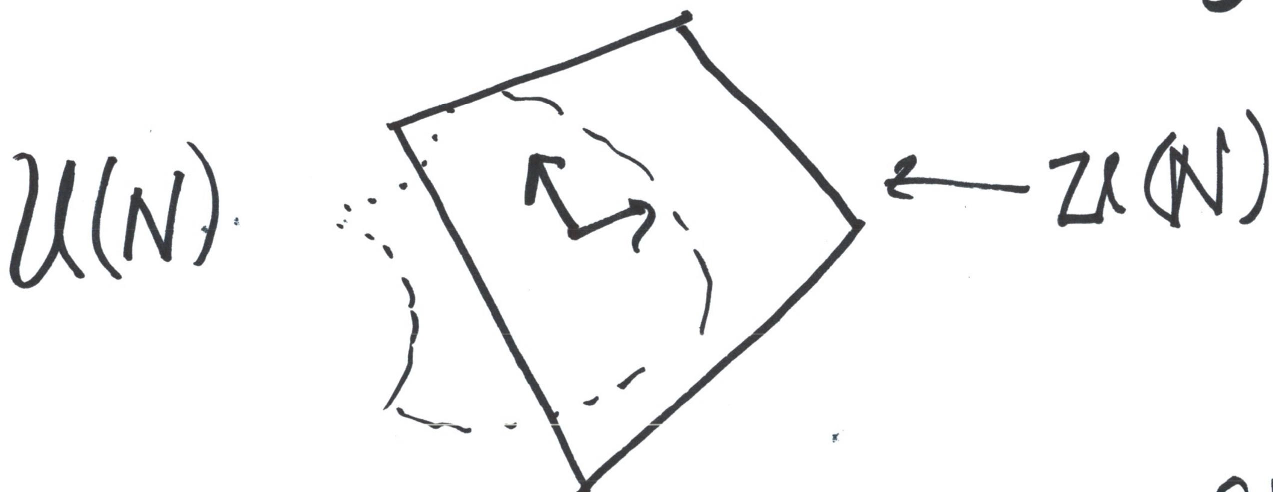
$$e^{tX} \in \mathcal{U}(N) \iff e^{-tX} = e^{tX^*} \ \forall t$$
$$\Rightarrow I - tX + O(t^2) = I + tX^* + O(t^2)$$
$$\Rightarrow -X = X^* \quad (t \to 0)$$

Other direction clear.

Intuition:



Lie group is a manifold,
Lie algebra is the tangent space at identity



Note: same ideas work for $G \leq GL(N, \mathbb{R})$,
with $\mathfrak{g} \subseteq Mat_{N\times N}(\mathbb{R})$.

<u>Ex</u>: $\mathrm{Lie}(O(N)) = \{X \in Mat_{N\times N}(\mathbb{R}) : X^T + X = 0\}$.

3. Haar measure near the identity

Let G be a matrix Lie group, and impose coordinates $(x_1, \dots, x_m)$ on vector space $\mathfrak{g}$.

Let μ be Haar measure on G, and $dx_1 \cdots dx_m$ Lebesgue measure $\mathfrak{g} \cong \mathbb{R}^m$.

Let $B_\varepsilon(I) = \{ g \in G : \|g - I\| < \varepsilon \}$

<u>Claim</u>: There is a constant c such that for small ε and any open $V \subseteq \mathfrak{g}$ with $\exp(V) \subseteq B_\varepsilon(I)$ (in G)

$$\mu(\exp(V)) = \left(c + o_{\varepsilon \to 0}(1)\right) \int_V dx_1 \cdots dx_m$$

uniformly in V.

Idea of proof:

- Map coordinates $(x_1, \ldots, x_m)$ to nbd of I in G, via exp map. Haar meas will be abs. cont. wrt push forward of Lebesgue, with continuous density function $M(x)$.

- Then the claim is generally true, for $\mu(dx_1 \ldots dx_m) = M(x)\, dx_1 \ldots dx_m$.