Arithmetic consequences of the GUE conjecture for zeta zeros

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Abstract. Conditioned on the Riemann hypothesis, we show that the conjecture that the zeros of the Riemann zeta function resemble the eigenvalues of a random matrix is logically equivalent to a statement about the distribution of primes.

This generalizes well known work that the pair correlation conjecture is equivalent to a statement about the variance of prime counts in short intervals and complements work of Farmer, Gonek, Lee, and Lester, who have considered similar questions conditioned on additional hypotheses which are not required here. As a byproduct of this argument, conditioned on the Riemann hypothesis we derive upper bounds for all moments of the logarithmic derivative of the Riemann zeta function.

We also discuss a conjecture for the covariance in short intervals of counts of almost-primes, weighted by the higher-order von Mangoldt function, and show the GUE Conjecture implies a weighted version of this conjecture. The covariance is surprisingly simple to write down.

Contents

Notation

$f(x) \lesssim g(x)$	There is a constant C such that $\lvert f(x)\rvert \le Cg(x)$. Used interchangeably with $f(x) = O(g(x))$.
$f_k(x) \lesssim_k g_k(x)$	There is a constant C_k depending on k so that $\lvert f_k(x)\rvert \le C_k g_k(x)$
$e(x)$	$e(x) = e^{i2\pi x\xi}$
$\hat{f}(\xi)$	$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e(-x\xi)\,dx$
$\check{f}(x)$	$\check{f}(x) := \int_{-\infty}^{\infty} f(\xi)e(x\xi)\,d\xi$
$\mathbb{N}_+$	$\mathbb{N}_+ := \{1, 2, 3, ...\}$
γ	imaginary ordinate of a nontrivial zeta zero, $\zeta(1/2 + i\gamma) = 0$
$C_c(\mathbb{R}^k)$	the set of continuous and compactly supported functions on $\mathbb{R}^n$
$\det_{n\times n}$	an $n \times n$ determinant
$K(x)$	$K(x) := \frac{\sin \pi x}{\pi x}$
$\Lambda(n)$	von Mangoldt function, $\log p$ if n is p^k the power of a prime, 0 otherwise
ψ	$\psi(x) := \sum_{n\le x} \Lambda(n)$
$\mathcal{U}(N)$	the group of $N \times N$ unitary matrices u, with Haar probability measure du
$Z(\beta)$	$Z(\beta) := \det(1 - e^{-\beta}u)$
$dz(x)$	$dz(x) := e^{-x/2}d\big(\psi(e^x) - e^x\big)$
admissible	See definition 2.3
$\upsilon_T(x, y)$	$\upsilon_T(x, y) := \big(1 - T\lvert x - y\rvert\big)_+$
Ψ_T	See equation (10)
Θ_T	See equation (11)
$\Lambda_j(n)$	$\Lambda_j(n) := \sum_{d\mid n} \mu(d)\log^k(n/d)$
ψ_j	$\psi_j(x) := \sum_{n\le x} \Lambda_j(n)$
$\tilde{\psi}_j$	See equation (22)
$\tilde{\psi}_j(x; H)$	$\tilde{\psi}_j(x; H) = \tilde{\psi}_j(x + H) - \tilde{\psi}_j(x)$
$H_j(r)$	See equations (79) and (80)
$S(t)$	$S(t) := \frac{1}{\pi} \arg \zeta(1/2 + it)$
$\Omega(t)$	$\Omega(t) := \frac{1}{2}\frac{\Gamma'}{\Gamma}\big(\frac{1}{4} + i\frac{t}{2}\big) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\big(\frac{1}{4} - i\frac{t}{2}\big) - \log \pi$
$\int_{\to-\infty}^{\to\infty}$	An improper integral
α, α_R	Bump functions centered at 0 of width 2 and $2R$; see equations (27), (28)
$Z_T, Z_T(\sigma)$	Point processes induced by zeta zeros, see definitions 5.1 and 5.4
GUE(σ)	The GUE Conjecture with averaging σ, see definition 5.5
$\mathcal{S}$	The sine-kernel determinantal point process (See Appendix B)
$\mathcal{S}'_N$	See definition 5.7
G_T	$G_T(\eta, t) := \sum_\gamma \eta\Big(\frac{\log T}{2\pi}(\gamma - t)\Big)$
L_T	$L_T(\eta, t) := \int_{-\infty}^{\infty} \eta\Big(\frac{\log T}{2\pi}(\xi - t)\Big) \frac{\log(\lvert\xi\rvert+2)}{2\pi}\, d\xi$
$\widetilde{G}_T$	$\widetilde{G}_T(\eta, t) := \int_{-\infty}^{\infty} \eta\Big(\frac{\log T}{2\pi}(\xi - t)\Big)\, dS(\xi)$
M_k	See equation (37)
$d\lambda_k(t)$	$d\lambda_k(t) := \log^k(\lvert t\rvert + 2)\, dt$
ω_ϵ	$\omega_\epsilon(x) := 1 - \alpha_\epsilon(x)$
Ω_ϵ	$\Omega_\epsilon(x) := \omega_\epsilon(x)\mathbf{1}_{\mathbb{R}_+}(x)$
$f\rvert_\epsilon$	$f\rvert_\epsilon(x) := f(x)\Omega_\epsilon(x)$
$f\rvert_a^b$	$f\rvert_a^b(x) := f\rvert_a(x) - f\rvert_b(x)$

1. Background material

1.1. We assume the Riemann hypothesis (RH) throughout this note.[1] Having assumed RH, we use the now standard notation of labeling the nontrivial zeros of the Riemann zeta function (with multiplicity[2]) by $1/2 + i\gamma$, with γ always a real number.

Recall that the GUE Conjecture for the high-lying zeros of the zeta function states that the local (or 'microscopic') spacing between the numbers γ resembles the local or microscopic spacing between the bulk eigenvalues of a random Hermitian matrix drawn from the Gaussian Unitary Ensemble [**3**]. More precisely,

Conjecture 1.1 (GUE). *For any fixed n and any fixed $\eta \in C_c(\mathbb{R}^n)$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \sum_{\substack{\gamma_1,\ldots,\gamma_n \\ \text{distinct}}} \eta\Big(\tfrac{\log T}{2\pi}(\gamma_1 - t), \ldots, \tfrac{\log T}{2\pi}(\gamma_n - t)\Big)\, dt = \int_{\mathbb{R}^n} \eta(x) \det_{n\times n}\Big(K(x_i - x_j)\Big)\, d^n x \quad (1)$$

where the entries of the $n \times n$ determinant are formed from the function $K(x) = \frac{\sin \pi x}{\pi x}$.

The sum on the left is over all collections of distinctly labeled ordinates $\gamma_1, \ldots, \gamma_n$. Though it should be noted, the compact support of test functions η means that our sums are effectively restricted to those γ's that are within $O(1/\log T)$ of the variable t.[3] It is natural to dilate the point $\gamma - t$ by a factor of $\log T/2\pi$ as we have here, since the γ's have density roughly $\log T/2\pi$ near $t \in [T, 2T]$.

The GUE Conjecture as stated above can be put somewhat more succinctly in probabilistic language: that the point processes induced by the stretched out zeta zeros converge in correlation to the determinantal point process with sine-kernel. We will define this terminology and elaborate on this point in section 5.

The terminology *GUE Conjecture* stems from the fact that the eigenvalues of a matrix from the Gaussian Unitary Ensemble will follow this limiting spacing in the bulk (see the surveys [**3**, Sec. 1] or [**16**, Sec. 6] for further details). This terminology is largely historical, since the GUE Conjecture involves only a comparison of zeta zeros to the limiting sine-kernel determinant, and a number of other random matrix ensembles have eigenvalues with this limiting distribution. For us it will be most convenient to compare the zeros of the zeta function to eigenvalues of large random unitary matrices (the so-called *Circular Unitary Ensemble*). See Section 2 for further details on this comparison.

The GUE Conjecture was first put forward in an important paper of Montgomery [**45**] on the basis of work he had done for the case $n = 2$ (see p. 184 of [**45**] for this discussion). [**45**] says only that one may expect the correlation functions of zeta zeros to tend to the sine-kernel determinants above, leaving it to the reader to determine how correlation functions of zero zeros ought to be defined. A common interpretation was given by [**55**]. We briefly discuss this interpretation of the GUE Conjecture and outline in Appendix C why it is equivalent to the formulation given above.

[1] In particular, while we explicitly preface all our main theorems with the label "On RH," this should be understood to apply even to smaller lemmata, although we will not include this explicit label in their statement.

[2] Labeling with multiplicity means that in a sum $\sum_\gamma$ if the zero corresponding to γ is not simple it appears more than once. (Of course, it is a classical conjecture that all zeros are simple, but we need not assume so.)

[3] We also note that the test functions η for which one may conjecture this relation can be extended to a slightly wider class. See Prop. 9.4 for a more technical discussion.

There is by now wide theoretical and numerical evidence in favor of the GUE Conjecture. We know that it is true for $n = 1$ (this fact is just the statement that around T the zeros have density $\frac{\log T}{2\pi}$, and dates back in different language to Riemann's memoir [**51**]). For $n \geq 2$ we can verify equation (1) if η is restricted to a (stringently) smooth class of test functions, a set of results initiated by Montgomery's seminal work [**45**] in the $n = 2$ pair correlation case, followed by work of Hejhal [**30**] and Rudnick-Sarnak [**55**] for the $n = 3$ and $n \geq 3$ cases respectively. Additionally the conjecture has by now a great deal of numerical support, beginning with the work of Odlyzko [**49**]. (The conjecture is sometimes known as the Montgomery-Odlyzko law for this reason. [**31**] is an account of numerical work.) Finally, we may mention a large number of analogous results which have been proven unconditionally in the function field setting, beginning with work of Katz and Sarnak [**37**], [**38**].

1.2. It is a matter of longstanding interest to see what can be said about the n-level correlation sums on the left hand side of (1) for functions not as smooth as those considered by Montgomery, Hejhal, and Rudnick & Sarnak once additional assumptions have been made about the distribution of the primes. Even in the original paper of Montgomery, the $n = 2$ pair correlation conjecture for a wider class of test functions was supported on the assumption of a uniform version of a the Hardy-Littlewood conjecture about the likelihood that two primes are separated by a small distance h. (This argument appears in [**46**].)

An especially relevant result in this direction is the following:

THEOREM 1.2 (Gallagher & Müller, and Goldston). *(On RH.) The $n = 2$ pair correlation conjecture is equivalent to the statement that for fixed $\beta \geq 1$, as $T \to \infty$,*

$$\int_1^{T^\beta} \Big(\psi\big(x + \tfrac{x}{T}\big) - \psi(x) - \tfrac{x}{T}\Big)^2 \frac{dx}{x^2} \sim (\beta - \tfrac{1}{2})\frac{\log^2 T}{T}. \tag{2}$$

The prime number theorem is a statement that the 'mean value' of $\psi(x)$ is x, so that this is a weighted estimate for the variance of the number of primes in short intervals $(x, x + x/T)$. That the pair correlation conjecture implies it is due to Gallagher and Mueller [**22**], the reverse implication to Goldston [**24**].

Unconditionally, for $\beta \leq 1$ the left hand side of (2) can be seen using the prime number theorem to be asymptotic to

$$\frac{\beta^2}{2}\frac{\log^2 T}{T}.$$

The somewhat unnatural weight dx/x^2 was removed in the work of Goldston and Montgomery [**27**], who showed that (on RH) a slightly stronger variant of the pair correlation conjecture is equivalent to a somewhat more naturally weighted estimate for the variance of primes in short intervals:

$$\frac{1}{X}\int_1^X \big(\psi(x+H) - \psi(x) - H\big)^2 \, dx \sim H\,(\log X - \log H) \tag{3}$$

uniformly for $1 \leq H \leq X^{1-\epsilon}$ (for any fixed $\epsilon > 0$). The survey [**25**] is a nice introduction to this and other material.

We mention that the counts $(\psi(x+H) - \psi(x) - H)$ for x a random variable uniformly distributed between 1 and large X are widely expected to be normally distributed with variance given by (3) (see [**47**]), though its higher moments are not directly related to the *local* statistics of zeros dealt with by Conjecture 1.1.

A computation reveals that neither (2) nor (3) are consistent with a heuristic model of Cramér [**11**] (see also [**28**], [**59**]) for the distribution of primes: that each number m has, roughly, an independent probability of $1/\log m$ of being prime. In these matters it is the predictions (2) and (3), rather than the Cramér model, that is widely expected to return the right answer. The Cramér model accurately predicts the Riemann hypothesis prediction that the error term in a count of primes in the interval $[1,x]$ is $O(x^{1/2+\epsilon})$, but quite apparently to accurately answer asymptotic questions about the distribution of primes in shorter intervals $[x, x+H]$ one must use a model of the primes that takes into account local arithmetic considerations.

Indeed, for higher correlations, Bogomolny and Keating [**1**], [**2**] argued heuristically that the m-level correlations correspond arithmetically to the likelihood that products of primes $p_1 \cdots p_\ell$ (each prime chosen from a specified region) are separated by a small distance from products of primes $p_{\ell+1}\cdots p_m$ (again with each prime drawn from a specified region) and that this likelihood – and therefore the GUE Conjecture – can be understood as before by using Hardy-Littlewood conjectures. These predict the probability in terms of a, b, and h that both p_1 and p_2 are prime, given that $ap_1 - bp_2 = h$, where p_1 and p_2 are of order x. The prediction is not $1/\log^2 x$, as one might guess from a naïve use of the Cramér model.

1.3. It is thus a matter of longstanding interest to generalize the work mentioned above in for instance Theorem 1.2 from the pair correlation conjecture to higher order correlations, and this is the purpose of the present paper.

During the time this work was in progress, a paper of Farmer, Gonek, Lee and Lester [**18**] addressed a closely related matter. Conditioned in addition to RH on technical hypotheses about the zeta zeros which they define and label Hypothesis AC and Hypothesis LC, the authors arrive at a solution in one direction, showing that knowing a Fourier-transformed evaluation of the n-point correlation sums in (1) (the *n-level form factor*) is sufficient to estimate the likelihood that products of primes in the fashion of Bogomolny and Keating are close to other products of primes.

Additionally motivated by the work of Goldston, Gonek, and Montgomery [**26**], the authors show conditioned on RH and Hypotheses AC and LC that knowing the n-level form factor for all n is sufficient to asymptotically evaluate

$$\frac{1}{T}\int_T^{2T} \prod_{\ell=1}^{j} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A_\ell}{\log T} + it\Big) \prod_{\ell'=1}^{k} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{B_{\ell'}}{\log T} - it\Big)\, dt \tag{4}$$

for positive constants $A_1, ..., A_j$, $B_1, ..., B_k$. Random matrix theory makes a prediction that this quantity will be asymptotic to a constant depending on the A_ℓ's and $B_{\ell'}$'s multiplied by $\log^{j+k} T$. Moreover one can proceed in the converse direction: the GUE Conjecture follows from a conjectured asymptotic evaluation of (4) for all j, k. That the pair correlation conjecture follows in this way from just the $j = k = 1$ case appears in [**26**], while the more general case follows from [**9**]

Finally, Farmer, Gonek, Lee, and Lester show that by assuming Hypothesis AC and LC in addition to RH one can bound for any fixed $A > 0$,

$$\frac{1}{T}\int_T^{2T} \left|\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big)\right|^k dt \lesssim \log^k T. \tag{5}$$

By Hölder this implies (4) is bounded by $O(\log^{j+k} T)$. On RH, the authors note, this is a correct lower bound.

The estimate (5) was also considered by Farmer and Ki [**19**], who also produce a bound, conditioned on RH and additional assumptions regarding the distribution of zeros.

1.4. The purpose of this paper is to demonstrate (on RH) that the GUE Conjecture as stated in Conjecture 1.1 not only implies but is in fact equivalent to a statement about the distribution of primes; this is the content of Theorem 2.4 as well as Theorem 13.2. Furthermore in Theorem 2.2 we show that the GUE Conjecture is equivalent to an asymptotic evaluation of (4). We do not require the Hypothesis AC or LC for this.

We make use of a somewhat different Tauberian technique than has been used in the past in the study of these problems. In addition this paper introduces the language of point processes to study the problems listed here, which may be of independent interest.

Our techniques additionally yield (5) on the assumption of RH but no other hypothesis.

The work makes it possible to restate for instance the $k = 3, 4$ triple and quadruple correlation conjectures for the zeta zeros in terms of the distribution of prime numbers. Unfortunately the resulting statements about the primes are complicated algebraically. We note however that a consequence of the GUE Conjecture is an estimate for the covariance of almost-primes in short intervals which is pleasant to state, where almost primes are weighted by the higher von Mangoldt functions famously used by Selberg and Erdős in proofs of the prime number theorem [**57**],[**14**].

2. A statement of main results

2.1. We obtain in the first place,

Theorem 2.1. *(On RH.) For fixed $k \geq 1$ and constant A with $\Re A > 0$,*

$$\frac{1}{T}\int_T^{2T} \left| \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big)\right|^k dt \lesssim_{A,k} \log^k T.$$

Remark: As noted in [**18**] one can obtain this for $A \geq 4$ by using Lemma 3 of Selberg's paper [**56**]. In fact, using instead Lemma 2 of Selberg's paper together with an upper bound due to Fujii (see Theorem 7.2), one can obtain exactly this theorem, for A arbitrarily close to 0 as above. We give a proof of Theorem 2.1 independent of Selberg's identity, since this will at any rate fit naturally into our framework, though we outline what the approach through Selberg's identity would look like. In some sense any possible proof must hinge upon the same ideas.

With sufficient effort one can trace through the implied constant in Theorem 2.1 in terms of A and k, obtaining a constant for positive real A of order

$$A^{-k}e^{O(k\log k)}$$

One should not expect this to be an optimal constant, or even necessarily the limit to which analysis on RH can be applied, though we do not pursue the matter further.

Indeed, for fixed $A > 0$ and positive integer λ, by assuming the GUE Conjecture one can show

$$\frac{1}{T}\int_T^{2T} \left| \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big)\right|^{2\lambda} dt \sim C(A, 2\lambda)\log^{2\lambda} T \tag{6}$$

where

$$C(A,2\lambda) := \lim_{N\to\infty} \frac{1}{N^{2\lambda}} \int_{\mathcal{U}(N)} \left|\frac{Z'}{Z}\Big(\frac{A}{N}\Big)\right|^{2\lambda} du,$$

$\mathcal{U}(N)$ is the group of $N \times N$ unitary matrices u with Haar probability measure du, and

$$Z(\beta) := \det(1 - e^{-\beta} u).$$

Note that if $\omega_1, ..., \omega_n$ are the eigenvalues of the unitary matrix u,

$$\frac{Z'}{Z}(\beta) = \sum_i \frac{1}{1 - e^{-\beta}\omega_i} = \sum_{r=1}^{\infty} e^{-\beta r}\mathrm{Tr}(u^r). \tag{7}$$

That the limit $X(A, 2\lambda)$ exists can be seen from the proof of Theorem 2.2 to follow. By computation with correlation functions, not reproduced here, one can see that for fixed λ, $C(A, 2\lambda)$ is of order $A^{-2\lambda+1}$ which for small A is slightly better than what can be obtained without refining our methods. (Though note for $\lambda = 1$ this order of bound is achieved in [**26**].)

2.2. It is by only slightly extending (6) that one can obtain a statement equivalent to the GUE Conjecture.

THEOREM 2.2. *(On RH.) The GUE Conjecture is equivalent to the statement that for all fixed $j, k \geq 1$ and all collections of fixed constants $A_1, ..., A_j, B_1, ..., B_k$ each with positive real part, the limit*

$$\lim_{T\to\infty} \frac{1}{\log^{j+k} T} \left(\frac{1}{T} \int_T^{2T} \prod_{\ell=1}^{j} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A_\ell}{\log T} + it\Big) \prod_{\ell'=1}^{k} \overline{\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{B_{\ell'}}{\log T} + it\Big)}\, dt \right) \tag{8}$$

exists and is equal to

$$\lim_{N\to\infty} \frac{1}{N^{j+k}} \left(\int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} \frac{Z'}{Z}\Big(\frac{A_\ell}{N}\Big) \prod_{\ell'=1}^{k} \overline{\frac{Z'}{Z}\Big(\frac{B_{\ell'}}{N}\Big)}\, du \right). \tag{9}$$

Moreover, for each $n \geq 1$, the claim that identity (1) *holds for all $k \leq n$ (that is, the zeros k-level correlation functions tend to that of the sine-kernel determinantal point process), is equivalent to the claim that these limits are equal for all $j + k \leq n$.*

It has long been understood in a heuristic sense that the characteristic polynomial Z is statistically an analogue of the zeta-function. (See [**41**] for the first spectacular application of this philosophy). Theorem 2.2 may be thought of as saying that at the microscopic scale described by the GUE Conjecture this correspondence should be understood quite literally. Theorem 2.2 may be compared with results in [**26**] and more recently [**6**] drawing an equivalence to the pair correlation conjecture. Note also work of the author [**53**] regarding ratios of the zeta function.

2.3. A theorem in the same spirit restates the GUE Conjecture in purely arithmetical terms.

To state the theorem more succinctly we require the notation

$$dz(x) := e^{-x/2}\, d\big(\psi(e^x) - e^x\big),$$

a measure which (because of its discrete part and growth as $|x| \to \infty$) we will only integrate against functions $\phi(x)$ that belong to a restricted class we call *admissible*:

DEFINITION 2.3. *A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is admissible if it is in $C^2(\mathbb{R})$, equal to 0 for sufficiently large x as $x \to \infty$, and bounded by $e^{\alpha|x|}$ for $\alpha < 1/2$ as $x \to -\infty$.*

If ϕ is admissible,

$$\int_{\mathbb{R}} \phi(x)\, dz(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(\log n) - \int_0^\infty \frac{1}{\sqrt{t}} \phi(\log t)\, dt,$$

which is a count of primes minus a regular approximation to that count.

Remark: By making use of improper integrals, in section 4 we will slightly extend the range of functions against which dz may be integrated, but any instance in which this extended definition is used will be made clear.

To reduce the length of formulas, we set

$$\upsilon_T(x,y) := \big(1 - T|x-y|\big)_+,$$

which plays the role of telling us when x and y are separated by a distance of $O(1/T)$.

Finally for bounded functions $f \in C^2(\mathbb{R}^j)$ and $g \in C^2(\mathbb{R}^k)$ such that $f \cdot \mathbf{1}_{\mathbb{R}^j_+}$ and $g \cdot \mathbf{1}_{\mathbb{R}^k_+}$ are compactly supported we define the arithmetical quantity

$$\begin{aligned} \Psi_T(f;g) &= \Psi_T^{j,k}(f;g) \qquad (10)\\ &:= \frac{1}{\log^{j+k} T} \int_{\mathbb{R}^j} \int_{\mathbb{R}^k} f\big(\tfrac{x}{\log T}\big) g\big(\tfrac{y}{\log T}\big) \upsilon_T(x_1 + \cdots + x_j\,,\, y_1 + \cdots + y_k)\, dz^k(y)\, dz^j(x). \end{aligned}$$

In the definition (10) for Ψ_T we will see later that the values f and g outside the quadrants $\mathbb{R}^j_+$ and $\mathbb{R}^k_+$ play no role asymptotically. Nonetheless, in (10) there is a certain algebraic significance to retaining integrals over all $\mathbb{R}^j \times \mathbb{R}^k$ rather than restricting to only this quadrant.

We likewise define the random matrix quantity

$$\begin{aligned} \Theta_N(f;g) &= \Theta_N^{j,k}(f;g) \qquad (11)\\ &:= \frac{1}{N^{j+k}} \sum_{r \in \mathbb{N}^j_+} \sum_{s \in \mathbb{N}^k_+} f\Big(\frac{r}{N}\Big) g\Big(\frac{s}{N}\Big) \int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} (-\mathrm{Tr}\, u^{r_\ell}) \prod_{\ell'=1}^{k} \overline{(-\mathrm{Tr}\, u^{s_{\ell'}})}\, du, \end{aligned}$$

As before, it is not immediately obvious that $\Theta_N(f;g)$ has a limiting value as $N \to \infty$ but we demonstrate this later.

THEOREM 2.4. *(On RH.) The GUE Conjecture is equivalent to the statement that for all fixed $j,k \geq 1$, and all collections of fixed collections of admissible functions $f_1, ..., f_j, g_1, ..., g_k$, we have for $f = f_1 \otimes \cdots \otimes f_j$ and $g = g_1 \otimes \cdots \otimes g_k$*

$$\lim_{T\to\infty} \Psi_T(f\,;\,g) = \lim_{N\to\infty} \Theta_N(f\,;\,g). \qquad (12)$$

Moreover, for each $n \geq 1$, the claim that identity (1) holds for all $k \leq n$ (that is, the zeros k-level correlation functions tend to that of the sine-kernel determinantal point process), is equivalent to the claim that (12) holds for all $j + k \leq n$.

Remark: $f_1 \otimes \cdots \otimes f_j$ is just the function $(x_1, ..., x_j) \mapsto f_1(x_1) \cdots f_j(x_j)$. Though it is only a technical point, in our proof it is important that the functions in (12) are separable. To have a simple proof which extends to non-separable functions would be desirable. Morally, the reason that separable functions by themselves are sufficient to recover the GUE Conjecture is that (12) is a linear relation, and linear combinations of such functions are sufficiently dense to approximate an arbitrary function. The same holds true for test functions $\exp(-A_1x_1 - \cdots - A_jx_j - B_1y_1 - \cdots - B_ky_k)$ in Theorem 2.2.

2.4. It is worthwhile to see that Theorem 2.4 generalizes Theorem 1.2, in particular that it implies identity (2). We do so heuristically for the moment, with a more rigorous development to follow later.

We know that the $n = 1$, 1-level density, case of the GUE Conjecture is true. It therefore follows from Theorem 2.4 that the pair correlation conjecture is equivalent to the claim that for all $f, g \in C_c^2(\mathbb{R})$,

$$\lim_{T\to\infty} \frac{1}{\log^2 T} \int_{\mathbb{R}}\int_{\mathbb{R}} f\Big(\tfrac{x}{\log T}\Big) g\big(\tfrac{y}{\log T}\Big) \upsilon_T(x,y)\, dz(x)\, dz(y) \tag{13}$$

is equal to

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} f\Big(\frac{r}{N}\Big) g\Big(\frac{s}{N}\Big) \int_{\mathcal{U}(N)} \operatorname{Tr} u^r\, \overline{\operatorname{Tr} u^s}\, du. \tag{14}$$

We specialize to the case in which $f = g$ with both functions an arbitrarily close approximation to the characteristic function $\mathbf{1}_{[0,\beta)}$. In this way, choosing better and better approximations, one can see that the pair correlation conjecture implies that for all $\beta > 0$,

$$\lim_{T\to\infty} \frac{1}{\log^2 T} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{[0,\beta)}\Big(\tfrac{x}{\log T}\Big) \mathbf{1}_{[0,\beta)}\big(\tfrac{y}{\log T}\Big) \upsilon_T(x,y)\, dz(x)\, dz(y) \tag{15}$$

is equal to

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \mathbf{1}_{[0,\beta)}\Big(\frac{r}{N}\Big) \mathbf{1}_{[0,\beta)}\Big(\frac{s}{N}\Big) \int_{\mathcal{U}(N)} \operatorname{Tr} u^r\, \overline{\operatorname{Tr} u^s}\, du. \tag{16}$$

In fact, with a little more work – using the fact that $\upsilon_T(x,y)$ constrains $x \approx y$ in (13) and (15) – one can see that (15) for all $\beta > 0$ is sufficient to recover (13) for general f and g; but we leave details of this argument to the reader.

To see that (15) provides the same information as (2) note that

$$\upsilon_T(x,y) = T \int \mathbf{1}_{[x-1/T,x]}(t)\mathbf{1}_{[y-1/T,y]}(t)\, dt$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{[0,\beta)}\Big(\tfrac{x}{\log T}\Big)\mathbf{1}_{[0,\beta)}\Big(\tfrac{y}{\log T}\Big)\upsilon_T(x,y)\, dz(x)\, dz(y)\\
&= T\int_{\mathbb{R}}\int_0^{\beta\log T}\int_0^{\beta\log T} \mathbf{1}_{[x-1/T]}(t)\mathbf{1}_{[y-1/T,y]}(t)\, dz(x)dz(y)\, dt\\
&\sim T\int_0^{\beta\log T} \Big(\int_t^{t+1/T} dz(x)\Big)\Big(\int_t^{t+1/T} dz(y)\Big)\, dt\\
&\sim T\int_0^{\beta\log T} e^{-t}\Big(\int_t^{t+1/T} d\big(\psi(e^x)-e^x\big)\Big)^2\, dt\\
&\sim T\int_1^{T^\beta} \big(\psi(\tau e^{1/T}) - \psi(\tau) - (e^{1/T}-1)\tau\big)^2 \frac{d\tau}{\tau^2}\\
&\sim T\int_1^{T^\beta} \Big(\psi\Big(\tau + \tfrac{\tau}{T}\Big) - \psi(\tau) - \tfrac{\tau}{T}\Big)^2 \frac{d\tau}{\tau^2}.
\end{aligned}$$

Our purpose at the moment is only to reassure the reader that the quantities we are working with are meaningful, so we have not made the effort to rigorously justify our passage from expression to expression. Rigorous justification is provided in a more general context in section 13. (None of the steps involve anything more involved than a straightforward bounding of error terms.)

On the other hand, to evaluate (16) we make use of the well known identity (see for instance [**13**]) that for $r \geq 1$,

$$\int_{\mathcal{U}(N)} \operatorname{Tr} u^r \, \overline{\operatorname{Tr} u^s} \, du = \delta_{rs} \, r \wedge N. \tag{17}$$

(Here, recall the notation $r \wedge N$ to denote the minimum of r and N.) Hence (16) is given by

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{r \leq N^\beta} r \wedge N = \beta - 1/2$$

for $\beta \geq 1$. For $\beta < 1$ this limit is $\beta^2/2$.

The equality of (15) and (16) then, for $\beta \geq 1$ (the range of β for which we cannot simply evaluate (15) unconditionally from the prime number theorem), is exactly equation (2).

2.5. It is possible in this way to draw out arithmetical equivalences for the $k = 3, 4$ three and four point correlation conjectures for zeta zeros. The resulting arithmetical statements do not, however, have the simplicity of Theorem 1.2. We record them in Theorems 11.2 and 11.3.

2.6. At the same time, it is possible using Theorems 2.4 and 2.2 to generalize Theorem 1.2 in a way that is algebraically simple – though the compromise we suffer is that the result we shall now state is not equivalent to the GUE Conjecture, but is only a consequence of it.

We will require the higher-order von Mangoldt functions Λ_j, defined in the usual manner by

$$\Lambda_j(n) := \mu \star (\log^j)(n) = \sum_{d|n} \mu(d) \log^k(n/d) \tag{18}$$

or equivalently inductively by

$$\Lambda_j(n) = \Lambda \star \Lambda_{j-1}(n) + \log(n)\Lambda_{j-1}(n), \tag{19}$$

where $\Lambda_1 = \Lambda$, the usual von Mangoldt function, and we have used $\star$ to denote multiplicative convolution on the integers. This inductive definition makes clear that Λ_j is supported on integers with no more than j distinct prime factors. We likewise define

$$\psi_j(x) := \sum_{n \leq x} \Lambda_j(n). \tag{20}$$

The properties of ψ_j are discussed in greater length in Appendix A. Unconditionally, from residue calculus and well-known zero-free regions for the zeta function, we know that

$$\begin{aligned} \psi_j(x) &= \operatorname*{Res}_{s=1} (-1)^j \frac{\zeta^{(j)}(s)}{\zeta(s)} \frac{x^s}{s} + o(x) \\ &= x P_{j-1}(\log x) + o(x), \end{aligned} \tag{21}$$

where $P_{j-1}(x)$, defined by this expression, is a $j-1$ degree polynomial with

$$P_{j-1}(\log x) = j \log^{j-1} x + o(\log^{j-1} x).$$

The error term between ψ_j and its regular approximation,

$$\tilde{\psi}_j(x) := \psi_j(x) - xP_{j-1}(\log x), \tag{22}$$

on Riemann hypothesis has the better bound, $O_j(x^{1/2+\epsilon})$, and finally we define

$$\tilde{\psi}_j(x;H) = \tilde{\psi}_j(x+H) - \tilde{\psi}_j(x),$$

which is a count of almost primes in an interval of length H, minus its regular approximation. Its regular approximation should be thought of as its 'expected value.'

We can arrive at counts of almost primes with the above von Mangoldt weights by repeatedly convolving the measures dz with one another, and in this way we will obtain

THEOREM 2.5. *(On RH.) On the assumption of the GUE Conjecture, for fixed $\beta > 0$ and integers $j,k \geq 1$, let $X = T^\beta$ and $\delta = 1/T$. Then*

$$\int_1^X \tilde{\psi}_j(x;\,\delta x)\tilde{\psi}_k(x,\,\delta x)\frac{dx}{x^2} \sim \tfrac{jk}{j+k-1}\frac{\log^{j+k} T}{T}\int_0^\beta y^{j+k-1}\wedge 1\,dy. \tag{23}$$

It is perhaps more instructive to write the right hand side of (23) as

$$\tfrac{jk}{j+k-1}\int_1^{T^\beta}\Big(\frac{x}{T}\Big)\Big(\log(x) - \log\big(\tfrac{x}{T}\vee 1\big)\Big)^{j+k-1}\frac{dx}{x^2}.$$

Recalling that $\delta x = x/T$ and $X = T^\beta$ above, it is reasonable therefore to make a conjecture in which the weight dx/x^2 has been replaced by the more natural weight dx.

CONJECTURE 2.6. *Fix any $\epsilon > 0$ and integers $j,k \geq 1$. Then as $X \to \infty$, uniformly for $1 \leq H \leq X^{1-\epsilon}$,*

$$\frac{1}{X}\int_1^X \tilde{\psi}_j(x;H)\tilde{\psi}_k(x;H)\,dx \sim \tfrac{jk}{j+k-1}\,H\,(\log X - \log H)^{j+k-1}. \tag{24}$$

By a summability argument, this agrees with (23).

Note that for $j = k = 1$ (24) agrees with estimate (3).

An elementary combinatorial computation applied to the prime number theorem will reveal that

$$\frac{1}{X}\sum_{n\leq X}\Lambda_j(n)\Lambda_k(n) \sim \frac{jk}{j+k-1}\log^{j+k-1} X, \tag{25}$$

and from this one may see that (23) is true unconditionally for $\beta \leq 1$, or alternatively that

$$\frac{1}{X}\int_1^X \tilde{\psi}_j(x;H)\tilde{\psi}_k(x;H)\,dx \sim \tfrac{jk}{j+k-1}\,H\,\log^{j+k-1} X.$$

for $H \leq 1$.

It is worth noting that in the function field setting in the limit of a large field size, an analogue of Conjecture 2.6 was proved by the author [**54**]. That proof was motivated by this conjecture and used equidistribution results of Katz [**39**] and a technique of Keating & Rudnick [**40**].

It would be interesting to better understand the arithmetical reasons that the diagonal contribution (25) so strongly determines the form of equations (23) and (24). Analogous results for the counts $\Lambda\star\cdots\star\Lambda$, for instance, in place of Λ_j and Λ_k can be derived from the GUE Conjecture, but do not have nearly so simple a form as j and k grow.

2.7. One can consider a more general class of von Mangoldt -type weights for almost primes than Λ_j and obtain an arithmetic statement involving covariance of almost primes which is equivalent to the GUE Conjecture. This is done in Theorem 13.2. However what is obtained is not as simple as Conjecture 2.6.

3. Acknowledgements

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4. Explicit formulae

In this section we introduce the explicit formulae relating the zeros to primes and the primes to the zeta function in the critical strip. In the forms we state these formulae, they are true only on RH. Unconditional formulations exist however.

The first of these is due in various stages to Riemann [**51**], Guinand [**23**], and Weil [**63**]. To state it in a notation that will be convenient for us, we need the classical notation

$$S(t) := \frac{1}{\pi} \arg \zeta(\tfrac{1}{2} + it)$$

with argument defined by a continuous rectangular path from 2 to $2+it$ to $1/2+it$, starting with $\arg \zeta(2) = 0$. For us, the importance of the function $S(t)$ is that on the Riemann hypothesis,

$$dS(t) = \Big(\sum_{\gamma} \delta_\gamma(t) - \frac{\Omega(t)}{2\pi} \Big)\, dt,$$

where

$$\Omega(t) := \tfrac{1}{2} \tfrac{\Gamma'}{\Gamma}\big(\tfrac{1}{4} + i\tfrac{t}{2}\big) + \tfrac{1}{2} \tfrac{\Gamma'}{\Gamma}\big(\tfrac{1}{4} - i\tfrac{t}{2}\big) - \log \pi.$$

By Stirling's formula

$$\frac{\Omega(t)}{2\pi} = \frac{\log\big((|t|+2)/2\pi\big)}{2\pi} + O\Big(\frac{1}{|t|+2}\Big),$$

and $\Omega(t)/2\pi$ is a regular approximation to the atomic mass at the γ's. $S(t)$ may therefore be thought of as an error term of a regular approximation to the zero counting function.

Theorem 4.1 (The explicit formula). *(On RH.) For g a function in $C_c^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} \hat{g}\Big(\frac{\xi}{2\pi}\Big)\, dS(\xi) = -\int_{-\infty}^{\infty} \big(g(x) + g(-x)\big)\, dz(x).$$

For more general functions g so long as g has appropriate continuity and decay conditions, such a formula remains true, provided in some cases we interpret the left hand integral as a principal value. (See for instance, [**48**], Theorem 12.13, or for a statement more along the lines above [**52**], Theorem 8.) For g delimited as above, it follows from standard Fourier analysis that $\hat{g}$ decays quadratically or faster, so that the left hand integral converges absolutely (since the contribution of both the atomic mass of zeta zeros and the mass $\Omega(t)/2\pi\, dt$ on an interval $[\xi, \xi+1]$ is at most $O(\log(|\xi|+2))$).

A related identity we will make use of relates the measure dz to the values of the zeta function in the critical strip. As with Theorem 4.1 it is true only on the Riemann hypothesis.

THEOREM 4.2. *(On RH.) For* $\Re s \in (0, 1/2)$,

$$-\frac{\zeta'}{\zeta}(1/2+s) = \int_{\to -\infty}^{\to\infty} e^{-sx}\, dz(x). \tag{26}$$

We have used the notation $\int_{\to-\infty}^{\to\infty}$ to denote an improper integral. Earlier to avoid any possible confusion we restricted the range of functions against which the measure dz can be integrated, and for this reason our improper integral must be defined in the following way:

We define the cutoff-function α_R by

$$\alpha(x) := \exp\Big(1 - \frac{1}{1-x^4}\Big)\mathbf{1}_{[-1,1]}(x), \tag{27}$$

$$\alpha_R(x) := \alpha(x/R). \tag{28}$$

For us the important features of α_R are that it is supported in $[-R, R]$, has continuous second derivative, and $\alpha(0) = 1$.

We thus define for a function $f \in C^2(\mathbb{R})$

$$\int_{\to-\infty}^{\to\infty} f(x)\, dz(x) = \lim_{R\to\infty} \int_{-\infty}^{\infty} \alpha_R(x) f(x)\, dz(x)$$

when the limit exists.

Note that we require the Riemann hypothesis to ensure that the integral in (26) converges; having assumed RH, that it does so follows from partial integration.

PROOF OF THEOREM 4.2. Note that for $\Re s > 1$, (by dominated convergence for instance),

$$F(s) := \lim_{R\to\infty} \int_0^\infty \alpha_R(x) e^{-sx} d\big(\psi(e^x) - e^x\big) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}. \tag{29}$$

But for any $\epsilon > 0$, it is easy to see by partial integration that the limit defining $F(s)$ converges uniformly for $\Re s \geq 1/2 + \epsilon$. Hence by analytic continuation (29) remains valid for $\Re s > 1/2$. Yet for $\Re s < 1$,

$$\int_{-\infty}^0 e^{(1-s)x}\, dx = -\frac{1}{s-1},$$

and so for $\Re s \in (1/2, 1)$,

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \alpha_R(x) e^{-sx} d\big(\psi(e^x) - e^x\big) = \lim_{R\to\infty} \int_0^\infty \alpha_R(x) e^{-sx} d\big(\psi(e^x) - e^x\big) + \int_{-\infty}^0 e^{(1-s)x}\, dx$$
$$= -\frac{\zeta'}{\zeta}(s)$$

by substituting (29). This is (26) with $s + 1/2$ replaced by s. □

5. Notation: point processes and linear statistics

5.1. We recast the GUE Conjecture in the language of point processes, with a short introduction to point processes given in Appendix B. A more general introduction may be found in [**58**] or [**32**]. Those uncomfortable with the notion of a point process may be reassured that for us the processes defined below will just be an abbreviation allowing us to write formulas more succinctly and bring to mind the positivity of certain quantities. Even an intuitive understanding would suffice to translate these formulas into a more familiar form.

5.2.

Definition 5.1. *Let T be a large real number, t a random variable uniformly distributed on $[T, 2T]$. We define Z_T to be the point process with point configurations*

$$\{\tfrac{\log T}{2\pi}(\gamma - t)\},$$

where γ runs over all the ordinates of non-trivial zeros of the zeta function.

Thus if we label the point configurations of Z_T by $\{\xi_j\}$, the formalism expressed by this definition allows to write

$$\mathop{\mathbf{E}}_{Z_T} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(\xi_{j_1}, ..., \xi_{j_k}) = \frac{1}{T}\int_T^{2T} \sum_{\substack{\gamma_1,\ldots,\gamma_k \\ \text{distinct}}} \eta\Big(\tfrac{\log T}{2\pi}(\gamma_1 - t), ... \tfrac{\log T}{2\pi}(\gamma_k - t)\Big)\, dt,$$

for $\eta \in C_C(\mathbb{R}^k)$.

5.3.

Definition 5.2. *$\mathcal{S}$ is the determinantal point process with sine-kernel.*

As discussed in Appendix B, the process $\mathcal{S}$ is characterized by its correlation functions, which have the value,

$$\mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(x_{j_1}, ..., x_{j_k}) = \int_{\mathbb{R}^k} \eta(x) \det_{k\times k}\Big(K(x_i - x_j)\Big)\, d^k x,$$

Definition 5.3. *Suppose for each $T > 0$, X_T is a point process with configurations $\{\xi_j(T)\}$, and suppose X is a point process with configurations $\{x_j\}$. We say that the processes X_T tend in correlation to the process X if for all $k \geq 1$ and all $\eta \in C_c(\mathbb{R}^k)$,*

$$\mathop{\mathbf{E}}_{X_T} \sum_{\substack{j_1,\cdots,j_k \\ \text{distinct}}} \eta(\xi_{j_1}(T), ..., \xi_{j_k}(T)) \to \mathop{\mathbf{E}}_{X} \sum_{\substack{j_1,\cdots,j_k \\ \text{distinct}}} \eta(x_{j_1}, ..., x_{j_k}).$$

Thus under this definition, the GUE Conjecture is just the statement that the processes Z_T tend in correlation as $T \to \infty$ to the process $\mathcal{S}$.

Remark: There is another (more canonical) notion of convergence of point processes, that of convergence in distribution. (See [**35**, Ch. 16] for a general account.) It is the case that Z_T tending to $\mathcal{S}$ in correlation (the GUE Conjecture as stated above) is equivalent to Z_T tending to $\mathcal{S}$ in distribution, though this does depend on some special properties of the processes Z_T. The notion of convergence in distribution and this equivalence is treated in more detail in the recent paper [**12**]. That paper also shows that the GUE Conjecture may be formulated as in terms of the distribution of *spacings* between zeros, rather than in terms of correlation functions. We will not require this fact or any discussion of convergence in distribution for point processes in what follows however, and refer the reader to [**12**] for further details.

5.4. If the reader is unhappy with the concept of point processes he or she will lose none of the logical structure of the argument simply by substituting $\frac{1}{T}\int_T^{2T}\cdots dt$ and a sum over $\frac{\log T}{2\pi}(\gamma - t)$ anytime he or she sees $\mathbf{E}_{Z_T}$ and a sum over ξ_i, and likewise substituting determinantal integrals for the expected value of correlation sums over $\mathcal{S}$.

We quickly demonstrate the notational advantage of this device however: with it we can write

$$\mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^{3} \sum_i \eta_\ell(x_i)$$

instead of

$$\mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{i_1,i_2,i_3\\ \text{distinct}}} \eta_1(\xi_{i_1})\eta_2(\xi_{i_2})\eta_3(\xi_{i_3}) + \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{i_1,i_2\\ \text{distinct}}} \eta_1(\xi_{i_1})\eta_2(\xi_{i_1})\eta_3(\xi_{i_2}) + \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{i_1,i_2\\ \text{distinct}}} \eta_1(\xi_{i_1})\eta_2(\xi_{i_2})\eta_3(\xi_{i_1})$$
$$+ \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{i_1,i_2\\ \text{distinct}}} \eta_1(\xi_{i_1})\eta_2(\xi_{i_2})\eta_3(\xi_{i_2}) + \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{i_1} \eta_1(\xi_{i_1})\eta_2(\xi_{i_1})\eta_3(\xi_{i_1})$$
$$= \int_{\mathbb{R}^3} \eta_1(x_1)\eta_2(x_2)\eta_3(x_3) \det_{3\times 3}\Big(K(x_i - x_j)\Big)\, d^3x + \int_{\mathbb{R}^2} \eta_1(x_1)\eta_2(x_1)\eta_3(x_2) \det_{2\times 2}\Big(K(x_i - x_j)\Big)\, d^2x$$
$$+ \int_{\mathbb{R}^2} \eta_1(x_1)\eta_2(x_2)\eta_3(x_1) \det_{2\times 2}\Big(K(x_i - x_j)\Big)\, d^2x + \int_{\mathbb{R}^2} \eta_1(x_1)\eta_2(x_2)\eta_3(x_2) \det_{2\times 2}\Big(K(x_i - x_j)\Big)\, d^2x$$
$$+ \int_{\mathbb{R}} \eta_1(x_1)\eta_2(x_1)\eta_3(x_1)\, dx_1.$$

The reader should check these expressions are the same.

5.5. In what follows, we will be using Fourier analysis in connection with the explicit formula, and for this reason it will be useful to replace the averages

$$\frac{1}{T}\int_T^{2T} \cdots\, dt = \int_{\mathbb{R}} \frac{\mathbf{1}_{[1,2]}(t/T)}{T} \cdots\, dt$$

with

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \cdots\, dt,$$

for σ a more general function. We define

Definition 5.4. *The point process $Z_T(\sigma)$ for σ a measurable function on $\mathbb{R}$ of mass 1 is defined by the point configurations*

$$\{\tfrac{\log T}{2\pi}(\gamma - t)\},$$

parameterized by a real valued random variable t with density $\sigma(t/T)/T$.

Note that under this definition, $Z_T = Z_T(\mathbf{1}_{[1,2]})$.

Definition 5.5. *For σ a measurable function on $\mathbb{R}$ of mass 1, we give the label* GUE(σ) *to the proposition that the processes $Z_T(\sigma)$ tend in correlation as $T \to \infty$ to the process $\mathcal{S}$.*

That is, in the language of correlation functions, GUE(σ) is the statement that for any $\eta \in C_c(\mathbb{R}^k)$,

$$\mathop{\mathbf{E}}_{Z_T(\sigma)} \sum_{\substack{j_1,\ldots,j_k\\ \text{distinct}}} \eta(\xi_{j_1},\ldots,\xi_{j_k}) = \int_{\mathbb{R}^k} \eta(x)\cdot \det_{k\times k}\Big(K(x_i - x_j)\Big)\, d^kx + o(1)$$

as $T \to \infty$.

In fact, there is nothing especially canonical about our use of $C_c(\mathbb{R}^k)$ test functions. Any class of test functions which are sufficiently dense and decay rapidly enough will do. We arrive at a more formal statement of this fact in section 9, where its proof will follow more easily.

5.6. The eigenvalues of the unitary group, stretched out so as to have mean unit density, also converge to the process $\mathcal{S}$. This can be seen from the integration formula of Weyl, which when combined with a lemma of Gaudin gives an exact evaluation for the k-point correlation functions of the eigenvalues:

THEOREM 5.6 (The Weyl-Dyson-Gaudin integration formula). *Let $\{e(\theta_1), ..., e(\theta_N)\}$ be the eigenvalues of a random $N \times N$ unitary matrix u, distributed according to Haar measure du, with θ_j chosen to be in $[-1/2, 1/2)$ for all j, and define*

$$K_N(x) := \frac{\sin \pi x}{N \sin(\pi x/N)}.$$

Then for any $k \leq N$ and measurable $\eta \colon [-N/2, N/2)^k \to \mathbb{C}$,

$$\int_{\mathcal{U}(N)} \sum_{\substack{j_1,...,j_k \\ distinct}} \eta(N\theta_{j_1}, ..., N\theta_{j_k})\, du = \int_{[-N/2,N/2)^k} \eta(x_1, ..., x_k) \det_{k\times k} \Big(K_N(x_i - x_j) \Big)\, d^k x.$$

Note that $K_N(x) \to K(x)$ uniformly.

We can form a point process even closer to $\mathcal{S}$ by pulling back the quantities $\{N\theta_j\}$, so that they are repeated with period N:

DEFINITION 5.7. *The point process $\mathcal{S}'_N$ is defined by the point configurations*

$$\bigcup_{\nu \in \mathbb{Z}} \{N(\theta_1 + \nu), ..., N(\theta_N + \nu)\}$$

where $\theta_1, ..., \theta_N \in [-1/2, 1/2)$ are, as in the Weyl integration formula, such that $\{e(\theta_1), ..., e(\theta_N)\}$ are the eigenvalues of a random unitary matrix distributed according to Haar measure.

If we label the point configurations of $\mathcal{S}'_N$ by $\{x_j\}$, the the Weyl integration formula gives that for $\eta \colon \mathbb{R}^k \to \mathbb{R}$ is integrable,

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N} \sum_{\substack{j_1,...,j_k \\ \text{distinct}}} \eta(x_{j_1}, ..., x_{j_k}) = \int_{\mathbb{R}^k} \eta(x) \cdot \det_{k\times k} \Big(K_N(x_i - x_j) \Big)\, d^k x.$$

In particular,

PROPOSITION 5.8. *$\mathcal{S}'_N \to \mathcal{S}$ in correlation.*

Note that by Poisson summation for functions η which are, for instance, in $C_c^2(\mathbb{R})$,

$$\sum_{\nu \in \mathbb{Z}} \eta(N\theta + N\nu) = \sum_{r \in \mathbb{Z}} \frac{1}{N} \hat{\eta}\Big(\tfrac{r}{N}\Big) e(r\theta)$$

for all θ, so that for $\eta_1, ..., \eta_k$ of this sort,

$$\mathop{\mathbf{E}}_{S'_N} \prod_{\ell=1}^{k} \sum_i \eta_\ell(x_i) = \frac{1}{N^k} \sum_{r \in \mathbb{Z}^k} \hat{\eta}_1\Big(\tfrac{r_1}{N}\Big) \cdots \hat{\eta}_k\Big(\tfrac{r_k}{N}\Big) \int_{\mathcal{U}(N)} \prod_{\ell=1}^{k} \mathrm{Tr}(u^{r_\ell})\, du. \tag{30}$$

Note that $\hat{\eta_\ell}$ for each ℓ will in this case decay quadratically, and for fixed N, $\mathrm{Tr}(u^r)$ remains bounded, so there is no difficulty in swapping the order of summation and integration.

It is therefore by passing through the processes S'_N that we will arrive at sums like (11).

Because the mapping $u \mapsto u^{-1}$ preserves Haar measure,

PROPOSITION 5.9. *For $r \in \mathbb{Z}^k$,*

$$\int_{\mathcal{U}(N)} \prod_{\ell=1}^k \mathrm{Tr}(u^{r_\ell})\, du = \int_{\mathcal{U}(N)} \prod_{\ell=1}^k \mathrm{Tr}(u^{-r_\ell})\, du.$$

In particular, as the left hand side is the complex conjugate of the right hand side, the expressions above are real valued.

5.7. Finally we introduce notation for linear statistics as they depend on the variable t. The mixed moments of these quantities carry the same information as the correlation sums (1) of the GUE Conjecture.

We define (for functions η that decay quadratically)

$$G_T(\eta, t) := \sum_\gamma \eta\Big(\tfrac{\log T}{2\pi}(\gamma - t)\Big). \tag{31}$$

An approximation to this count is given by substituting an integral against $\log(|\xi|+2)/2\pi$ for the sum over zeros:

$$L_T(\eta, t) := \int_{-\infty}^{\infty} \eta\Big(\tfrac{\log T}{2\pi}(\xi - t)\Big)\, \frac{\log(|\xi|+2)}{2\pi}\, d\xi. \tag{32}$$

Note that, for η that decay quadratically,

$$L_T(\eta, t) = \tfrac{\log(|t|+2)}{\log T} \int_{-\infty}^{\infty} \eta(\alpha)\, d\alpha + O_\eta\Big(\frac{1}{\log T}\Big).$$

Finally, we define

$$\widetilde{G}_T(\eta, t) := \int_{-\infty}^{\infty} \eta\Big(\tfrac{\log T}{2\pi}(\xi - t)\Big)\, dS(\xi). \tag{33}$$

From Stirling's formula, for η that decay quadratically,

$$\begin{aligned}\widetilde{G}_T(\eta, t) &= G_T(\eta, t) - L_T(\eta, t) + O_\eta\Big(\frac{1}{\log T}\Big)\\ &= G_T(\eta, t) - \tfrac{\log(|t|+2)}{\log T} \int_{-\infty}^{\infty} \eta(\alpha)\, d\alpha + O_\eta\Big(\frac{1}{\log T}\Big),\end{aligned} \tag{34}$$

so that $\widetilde{G}_T(\eta, t)$ should be thought of as the linear statistic $G_T(\eta, t)$ minus its expected value.

Since we know unconditionally that the number of γ in any interval $[k, k+1)$ is at most $\log(|k|+2)$, we have

$$G_T(\eta, t) \lesssim \sum_{k \in \mathbb{Z}} \log(|k|+2) \max_{x \in [k, k+1)} \Big|\eta\Big(\tfrac{\log T}{2\pi}(x - t)\Big)\Big|, \tag{35}$$

with the same upper bound obviously holding for $L_T(\eta, t)$, and therefore $\widetilde{G}_T(\eta, t)$. A particular consequence of (35) that we will make use of later is that if

$$\eta(\xi) \lesssim 1/(1+\xi^2),$$

then

$$G_T(\eta, t) \lesssim \log(|t|+2),$$

and likewise for $L_T(\eta,t)$ and $\widetilde{G}_T(\eta,t)$.

The arithmetic significance of $\widetilde{G}_T(\eta,t)$ comes from the explicit formula:

PROPOSITION 5.10. *For $g \in C^2(\mathbb{R})$,*

$$\widetilde{G}_T(\hat{g},t) = \frac{-1}{\log T}\int_{-\infty}^{\infty}\Big(g\Big(\tfrac{x}{\log T}\Big)e^{ixt} + g\Big(\tfrac{-x}{\log T}\Big)e^{-ixt}\Big)\,dz(x). \tag{36}$$

It is worthwhile to see one example of the relation between the two notations introduced in this section. We have, for instance,

$$\mathop{\mathbf{E}}_{Z_T(\sigma)}\Big(\sum \eta(\xi_j)\Big)^k = \int_{\mathbb{R}} \frac{\sigma(t/T)}{T}\, G_T(\eta,t)^k\,dt.$$

6. A plan of the proof

6.1. We are now in a position to outline our proofs. A tool we will find absolutely essential is an upper bound on the moments of point counts in the process Z_T first proved by Fujii [**20**] and which may be approximately stated in the following way, that for fixed k, as long as t has been averaged over a long enough interval (with length of order T), the k^{th} moment of the count

$$\#(\tfrac{\log T}{2\pi}(\gamma - t) \in [A, A+k])$$

remains bounded, uniformly of the choice of A. In the language of point processes, this is to say the moments of counts of points inside coarse enough intervals can be bounded from above to the correct order.

This cannot be literally true as it has been stated, because for large enough A the density of γ around $t + 2\pi A/\log T$ will be larger than $\log T/2\pi$. A precise statement is that uniformly in a and for any $\epsilon > 0$,

$$\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T}\big|G_T(\eta,t)\big|^k\,dt \lesssim_k \int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T}\big|L_T(M_k\eta,t)\big|^k\,dt$$

for all $T \geq T_0$, where T_0 is a function only of ϵ. Here we have used the notation M_k, which denotes an upper bound on η made from characteristic functions of size k:

$$M_k\eta(\xi) := \sum_{\nu=-\infty}^{\infty} \mathbf{1}_{I_k(\nu)}(\xi)\cdot \sup_{I_k(\nu)}|\eta|. \tag{37}$$

Here for typographical reasons we denote the interval $[k\nu - k/2, k\nu + k/2)$ by $I_k(\nu)$. Recall that $L_T(\cdot,t)$ amounts to replacing the sum over γ in $G_T(\cdot,t)$ with a logarithmic mass that approximates this sum.

We also prove another upper bound which is considerably more subtle. This is that for a function g supported in an interval $[-X,X] \subset [-1/k,1/k]$ and bounded in modulus by a constant A,

$$\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T}\big|\widetilde{G}_T(\hat{g},t)\big|^k\,dt \lesssim_k A^k X^k, \tag{38}$$

for $T \geq T_0$, with T_0 a function of only ϵ and X.

This result should be surprising at first glance for the following reason: if $g(x) \in C^2(\mathbb{R})$ closely approximates the indicator function

$$\mathbf{1}_{[-\delta,\delta]}(x)$$

then $\hat{g}(\xi)$ will closely approximate the function

$$\frac{1}{\delta}\frac{\sin(\pi\xi/\delta)}{\pi\xi/\delta}.$$

In particular as g approachs $\mathbf{1}_{[-\delta,\delta]}$ (say uniformly), the $L^1(\mathbb{R})$ norm of $\hat{g}$ will grow arbitrarily large. Yet the naive approach to bounding the left hand side of (38), namely

$$\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{T}\left|\sum_\gamma \hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big) - \int_{-\infty}^{\infty}\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\frac{\Omega(\xi)}{2\pi}\,d\xi\right|^k dt$$

$$\lesssim \int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{T}\left(\sum_\gamma \left|\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\right| + \int_{-\infty}^{\infty}\left|\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\right|\frac{\Omega(\xi)}{2\pi}\,d\xi\right)^k dt$$

will be arbitrarily large as

$$\sum_\gamma \left|\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\right| \text{ and } \int_{-\infty}^{\infty}\left|\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\right|\frac{\Omega(\xi)}{2\pi}\,d\xi$$

will both be large for every t. Perhaps even more surprising, our claim is that the left hand side of (38) becomes smaller as δ becomes smaller.[4] This can only be seen by exploiting the cancellation that arises by subtracting from

$$\sum_\gamma \hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)$$

its regular approximation

$$\int_{-\infty}^{\infty}\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\frac{\Omega(\xi)}{2\pi}\,d\xi.$$

The situation is analogous to estimating

$$\sum_{k\in\mathbb{Z}} f(k) - \int_{\mathbb{R}} f(x)\,dx = \sum_{k\in\mathbb{Z}} f(k) - \hat{f}(0).$$

A naive bound on this quantity is $2\|f\|_{L_1}$, but in fact for functions that do not oscillate much the sum over $\mathbb{Z}$ is close to the integral over $\mathbb{R}$. By Poisson summation if $\hat{f}$ is supported in $(-1,1)$, this quantity is exactly 0.

It is these two upper bounds that take the place in our proof of the Hypothesis AC and LC in [**18**]. They are proven in section 7 using the explicit formula.

Analogous upper bounds may be proven for the average distribution of eigenvalues of the unitary group under Haar measure. This is the content of section 8.

6.2. In section 9, we make use of the first of these upper bounds for the zeros of the zeta function to show that, for averages weighted by σ_1 and σ_2, the statements GUE(σ_1) and GUE(σ_2) are equivalent. This is a Tauberian theorem. We expand upon the ideas involved in its proof in section 9.

[4]Suppose for instance that we should exploit some additional cancellation in the oscillating $\hat{g}$ by looking at

$$\left|\sum_\gamma \hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\right| \text{ and } \left|\int_{-\infty}^{\infty}\hat{g}\Big(\tfrac{\log T}{2\pi}(\gamma-t)\Big)\frac{\Omega(\xi)}{2\pi}\,d\xi\right|$$

instead. Even this refinement in insufficient to obtain a bound that decreases with δ.

6.3. With this equivalence between weights we can give a first heuristic approximation to what lies behind our proof. By adding in lower correlations, GUE(σ) may be seen to be equivalent to the claim that

$$\mathop{\mathbf{E}}_{Z_T(\sigma)} \prod_{\ell=1}^{n} \Big(\sum_j \eta_\ell(\xi_j) - \int_{-\infty}^{\infty} \eta_\ell(\alpha)\, d\alpha \Big) = \mathop{\mathbf{E}}_{S} \prod_{\ell=1}^{n} \Big(\sum_j \eta_\ell(x_j) - \int_{-\infty}^{\infty} \eta_\ell(\alpha)\, d\alpha \Big) + o(1) \quad (39)$$

for every $n \geq 1$ and every $\eta_1, ...,$ and η_n belonging to a class of functions sufficiently dense in $C_c(\mathbb{R})$ and with suitable continuity and decay conditions. But the left hand side of (39) asymptotically is just

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \prod_{\ell=1}^{n} \widetilde{G}_T(\eta_\ell, t)\, dt \\
&= \int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \prod_{\ell=1}^{n} \Big(\frac{1}{\log T} \int_{-\infty}^{\infty} \hat{\eta}_\ell\Big(\tfrac{x}{\log T}\Big) e^{ixt} + \hat{\eta}_\ell\Big(\tfrac{-x}{\log T}\Big) e^{-ixt}\, dz(x) \Big)\, dt \\
&= \frac{1}{\log^n T} \sum_{\varepsilon \in \{-1,1\}^n} \int_{\mathbb{R}^n} \hat{\eta}_1\Big(\tfrac{\varepsilon_1 x_1}{\log T}\Big) \cdots \hat{\eta}_n\Big(\tfrac{\varepsilon_n x_n}{\log T}\Big) \hat{\sigma}\big(\tfrac{T}{2\pi}(\varepsilon_1 x_1 + \cdots + \varepsilon_k x_k)\big)\, dz(x_1) \cdots dz(x_k),
\end{aligned} \quad (40)$$

with the last two equalities following simply from computation through the explicit formula and interchanging the order of integration. If we now select σ so that $\hat{\sigma}(y) = (1 - 2\pi|x|)_+$, the reader may check (40) is just a polarization of the quantities Ψ_T defined in (10).

On the other hand, the right hand side of (39) can be evaluated as the limiting case of random matix statistics which end up being a polarized form of (11). Since by taking linear combinations of the identity (12) in Theorem 2.4 one can recover the polarized from above, it is comparatively easy in this way to see that (12) implies the GUE Conjecture.

To show that the GUE Conjecture implies (12) requires more work. To first approximation, the argument is nothing more than setting $\eta_1 \otimes \cdots \otimes \eta_n$ in (39) so that $\hat{\eta_1} \otimes \cdots \hat{\eta_n}$ is restricted to a given quadrant $\mathbb{R}_+^j \times \mathbb{R}_-^k$ of $\mathbb{R}^n$, and such that in these quadrants each $\hat{\eta}_\ell$ has a sharp cutoff at the origin, say $\hat{\eta}_\ell(x) = \mathbf{1}_{R_{\varepsilon_\ell}} f_\ell(\varepsilon_\ell x)$ for functions f_ℓ admissible in the sense of Definition 2.3. In this way,

$$\begin{aligned}
&\frac{1}{\log^n T} \sum_{\varepsilon \in \{-1,1\}^n} \int_{\mathbb{R}^n} \hat{\eta}_1\Big(\tfrac{\epsilon_1 x_1}{\log T}\Big) \cdots \hat{\eta}_n\Big(\tfrac{\epsilon_n x_n}{\log T}\Big) \hat{\sigma}\big(\tfrac{T}{2\pi}(\varepsilon_1 x_1 + \cdots \varepsilon_n x_n)\big)\, d^n z(x) \\
&= \frac{1}{\log^n T} \int_{\mathbb{R}_+^j} \int_{\mathbb{R}_+^k} f_1\Big(\tfrac{x_1}{\log T}\Big) \cdots f_j\Big(\tfrac{x_j}{\log T}\Big) f_{j+1}\Big(\tfrac{x_{j+1}}{\log T}\Big) \cdots f_{j+k}\Big(\tfrac{x_{j+k}}{\log T}\Big) \\
&\qquad\qquad \times \hat{\sigma}\big(\tfrac{T}{2\pi}(x_1 + \cdots + x_j - x_{j+1} - \cdots - x_{j+k})\, d^n z(x) + o(1) \\
&= \frac{1}{\log^n T} \int_{\mathbb{R}^j} \int_{\mathbb{R}^k} f_1\Big(\tfrac{x_1}{\log T}\Big) \cdots f_j\Big(\tfrac{x_j}{\log T}\Big) f_{j+1}\Big(\tfrac{x_{j+1}}{\log T}\Big) \cdots f_{j+k}\Big(\tfrac{x_{j+k}}{\log T}\Big) \\
&\qquad\qquad \times \hat{\sigma}\big(\tfrac{T}{2\pi}(x_1 + \cdots + x_j - x_{j+1} - \cdots - x_{j+k})\, d^n z(x) + o(1).
\end{aligned}$$

The last line will follow by showing that the 'tails'

$$f\Big(\tfrac{x}{\log T}\Big)\, dz(x) = f\Big(\tfrac{x}{\log T}\Big) e^{x/2}\, dx, \qquad \text{for } x \leq 0$$

do not substantially contribute to these quantities asymptotically. We have thus recovered the terms Ψ_T.

This approach must be amended very substantially however, owing to the fact that for Fourier transforms $\hat{\eta}$ with a sharp cutoff at the origin, the original distribution η will in general not be integrable, and so the sums in (39) are not well defined.

We overcome the issue by choosing smooth functions $f_\ell|_{\epsilon_T}$ (depending upon T) that so closely approximate functions of sharp cutoff $f_\ell \cdot \mathbf{1}_{R_+}$ that we still have

$$\begin{aligned}
&\frac{1}{\log^n T}\int_{\mathbb{R}_+^j}\int_{\mathbb{R}_+^k} f_1\Big(\tfrac{x_1}{\log T}\Big)\cdots f_j\Big(\tfrac{x_j}{\log T}\Big) f_{j+1}\Big(\tfrac{x_{j+1}}{\log T}\Big)\cdots f_{j+k}\Big(\tfrac{x_{j+k}}{\log T}\Big)\\
&\qquad\qquad\qquad\qquad \times \hat{\sigma}\big(\tfrac{T}{2\pi}(x_1+\cdots+x_j-x_{j+1}-\cdots-x_{j+k}\big)\, d^n z(x)\\
&=\frac{1}{\log^n T}\int_{\mathbb{R}^j}\int_{\mathbb{R}^k} f_1|_{\epsilon_T}\Big(\tfrac{x_1}{\log T}\Big)\cdots f_j|_{\epsilon_T}\Big(\tfrac{x_j}{\log T}\Big) f_{j+1}|_{\epsilon_T}\Big(\tfrac{x_{j+1}}{\log T}\Big)\cdots f_{j+k}|_{\epsilon_T}\Big(\tfrac{x_{j+k}}{\log T}\Big)\\
&\qquad\qquad\qquad\qquad \times \hat{\sigma}\big(\tfrac{T}{2\pi}(x_1+\cdots+x_j-x_{j+1}-\cdots-x_{j+k}\big)\, d^n z(x) + o(1).
\end{aligned}$$

It will indeed be the case that for this to be true, the closeness of our approximation of $f_\ell|_{\epsilon_T}$ to $f_\ell \cdot \mathbf{1}_{\mathbb{R}_+}$ must increase with T. All the same, for any $\delta > 0$, we show that there is a fixed approximation $f|_\epsilon$ so that

$$\begin{aligned}
\Bigg|&\frac{1}{\log^n T}\int_{\mathbb{R}^n} f_1|_\epsilon\big(\tfrac{x_1}{\log T}\Big)\cdots f_n|_\epsilon\Big(\tfrac{x_n}{\log T}\Big)\hat{\sigma}\big(\tfrac{T}{2\pi}(x_1+\cdots-x_n)\big)\, d^n z(x) \qquad (41)\\
&-\frac{1}{\log^n T}\int_{\mathbb{R}^n} f_1|_{\epsilon_T}\big(\tfrac{x_1}{\log T}\Big)\cdots f_n|_{\epsilon_T}\Big(\tfrac{x_n}{\log T}\Big)\hat{\sigma}\big(\tfrac{T}{2\pi}(x_1+\cdots-x_n)\big)\, d^n z(x)\Bigg| < \delta.
\end{aligned}$$

Because the functions $f_\ell|_\epsilon$ closely approximate $f_\ell \cdot \mathbf{1}_{\mathbb{R}_\epsilon}$,

$$\frac{1}{\log^n T}\int_{\mathbb{R}^n} f_1|_\epsilon\big(\tfrac{x_1}{\log T}\Big)\cdots f_n|_\epsilon\Big(\tfrac{x_n}{\log T}\Big)\hat{\sigma}\big(\tfrac{T}{2\pi}(x_1+\cdots-x_n)\big)\, d^n z(x)$$

will be close to its polarization

$$\sum_{\varepsilon\in\{-1,1\}^n}\frac{1}{\log^n T}\int_{\mathbb{R}^n} f_1|_\epsilon\big(\tfrac{\varepsilon_1 x_1}{\log T}\Big)\cdots f_n|_\epsilon\Big(\tfrac{\varepsilon_n x_n}{\log T}\Big)\hat{\sigma}\big(\tfrac{T}{2\pi}(\varepsilon_1 x_1+\cdots-\varepsilon_n x_n)\big)\, d^n z(x).$$

This last quantity, because the functions $f_\ell|_\epsilon$ are fixed and smooth, can be evaluated on the GUE Conjecture by identity (40). It is a straightforward matter finally to show that the resulting answer agrees with that of Theorem 2.4.

Although (41) is intuitive enough, we have not really fully justified it. Its proof in section 11 is technical and is accomplished only via the upper bound (38) and what is sometimes referred to as a tensorization trick. (This tensorization trick is the reason we work with separable functions.) Note that it is natural to apply (38) here, as the functions $(f_\ell|_\epsilon - f_\ell|_{\epsilon_T})$ are supported in a small region around the origin.

It is through this same method, using (26) of Theorem (4.2) and the fact that linear combinations of function $\exp(-A_1x_1 - \cdots - A_nx_n)$ are sufficiently dense in $C_c(\mathbb{R}_+^n)$, that we arive at Theorem 2.2.

6.4. Theorem 2.1 is an application of the same method of decomposing test functions into parts with different Fourier support. Letting $f(x) = \exp(-Ax)$, Theorem 4.2 gives that

$$\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big) = \frac{1}{\log T}\int_{\to -\infty}^{\to\infty} f\Big(\tfrac{x}{\log T}\Big)e^{ixt}\, dz(x).$$

We decompose this into

$$O\Big(\tfrac{1}{\log T}\Big) + \frac{1}{\log T}\int_{\mathbb{R}} f|_{\epsilon_T}^{1/k}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x) + \frac{1}{\log T}\int_{\mathbb{R}} f|_{1/k}^{R_T}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x),$$

where $f|_{\epsilon_T}^{1/k}$ is a function supported in the the interval $[0, 1/k]$ and $f|_{1/k}^{R_T}$ is chosen so that

$$f|_{\epsilon_T}^{1/k} + f|_{1/k}^{R_T}$$

is a smooth compactly supported function (on an interval $[0, R_T]$ say) that closely approximates

$$f \cdot \mathbf{1}_{\mathbb{R}_+}.$$

Note that for fixed k, one should be able to (and indeed can) choose such functions $f|_{1/k}^{R_T}$ in a way that their second derivatives do not increase with T.

We have that

$$\begin{aligned}\frac{1}{T}\int_0^T \left|\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big)\right|^k dt \lesssim_k & \frac{1}{T}\int_0^T \left|\frac{1}{\log T}\int_{\mathbb{R}} f|_{\epsilon_T}^{1/k}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x)\right|^k dt \\ &+ \frac{1}{T}\int_0^T \left|\frac{1}{\log T}\int_{\mathbb{R}} f|_{1/k}^{R_T}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x)\right|^k dt \\ =&\frac{1}{T}\int_0^T \Big|\widetilde{G}_T\big((f|_{\epsilon_T}^{1/k})\hat{\ }, t\big)\Big|^k dt + \frac{1}{T}\int_0^T \Big|\widetilde{G}_T\big((f|_{1/k}^{R_T})\hat{\ }, t\big)\Big|^k dt.\end{aligned}$$

The first of these terms can be bounded by (38). For the second, note that $f_{1/k}^{R_T}$ does not have increasing first or second derivative, even as T increases (because the cutoff from the origin to $1/k$ does not change with T). Therefore $(f_{1/k}^{R_T})\hat{\ }$ will decay quickly enough, for all T, that an appropriate bound can be gained from the Fujii upper bound.

6.5. The final part of this paper concerns evaluating the covariance of almost primes. We weight the almost primes in such a way as to produce an algebraically nice answer. The algebraic part involves certain random matrix statistics discussed later. On the other hand, we can quickly outline how it is that one arrives at counts of almost primes from Theorems 2.4 and 2.2 by convolving the measure dz with itself, so that for instance,

$$dz * dz(x) + x\,dz(x) = e^{-x/2} d\big(\psi_2(x) - xP_1(x)\big)$$

where ψ_2 and P_1 are defined by (20) and (21). In perhaps more familiar language, this is just that for $\Re s > 1$,

$$\frac{\zeta''}{\zeta}(s) = \sum_n \frac{\Lambda_2(n)}{n^s}.$$

To convolve the measure dz with itself, we must replace the test functions $f_1 \otimes \cdots \otimes f_n$ in Theorem 2.4 with test functions $f(x_1, ..., x_n)$ that are constant on level sets of $x_1 + x_2 + \cdots + x_n$, for instance. The fastest route to such a replacement is by appealing to Theorem 2.2, but because the test functions $\exp(-Ax)$ are not compactly supported, this route entaills a few technical challenges. These are discussed in more detail in section 13.

7. Upper bounds for counts of zeros

In this section we reference several lemmas from the paper [**52**].

7.1. The following computational lemma will be useful for us

LEMMA 7.1. *Suppose we are given non-negative integrable σ of mass 1 such that $\hat{\sigma}$ has compact support, and suppose $g_1, ..., g_k$ are in $C_c^2(\mathbb{R})$ and satisfy* supp $g_\ell \subset [-\delta_\ell, \delta_\ell]$ *with $\delta_1 + \cdots + \delta_k = \Delta \leq 2$. Then there exists a T_0 depending only on Δ and the region in which $\hat{\sigma}$ is supported so that for $T \geq T_0$,*

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \prod_{\ell=1}^{k} \widetilde{G}_T(\,\hat{g}_\ell\,, t)\, dt = \left(\frac{-1}{\log T}\right)^k \sum_{n_1^{\epsilon_1} n_2^{\epsilon_2} \cdots n_k^{\epsilon_k} = 1} \prod_{\ell=1}^{k} \frac{\Lambda(n_\ell)}{\sqrt{n_\ell}} g_\ell\left(\tfrac{\epsilon_\ell \log n_\ell}{\log T}\right) \qquad (42)$$

$$+ O_k\left(\frac{1}{T^{1-\Delta/2}} \prod_{\ell=1}^{k} \frac{\|g_\ell\|_\infty}{\log T}\right),$$

where the sum is over all $n \in \mathbb{N}^k, \epsilon \in \{-1,1\}^k$ such that $n_1^{\epsilon_1} n_2^{\epsilon_2} \cdots n_k^{\epsilon_k} = 1$.

PROOF. See Lemma 11 of [**52**]. □

7.2. As a consequence, we show that for coarse enough counts, linear statistics of zeta zeros can rigorously be bounded above to the correct order. This is the first upper bound outlined in section 6.

LEMMA 7.2 (A Fujii-type upper bound). *For σ non-negative and integrable such that $\hat{\sigma}$ is compactly supported, there exists a T_0 depending only on the region in which $\hat{\sigma}$ is supported, so that for all $T \geq T_0$,*

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \prod_{j=1}^{k} G_T(\eta_j, t)\, dt = O_k\left(\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \prod_{j=1}^{k} L_T(M_k \eta_j, t)\, dt\right),$$

where the implied constant depends only upon k.

The upper bound M_k is defined in (37).

Remark: Note that in the notation of point processes, the left hand side is

$$\mathop{\mathbf{E}}_{Z_t(\sigma)} \prod_{\ell=1}^{k} \sum_i \eta_\ell(\xi_i).$$

PROOF. See that of Lemma 15 in [**52**]. □

This is a slight generalization of an upper bound which Fujii proved using Selberg's mollification for the zeta function [**20**]. It is true for any functions $\eta_1, ...\eta_\ell$ but of course not meaningful unless the right hand side is finite.

We can state the lemma in more intuitive terms.

LEMMA 7.3 (A Fujii-type upper bound, restated). *For $\epsilon_0 > 0$, there exists a T_0 depending only on ϵ_0 so that for all $a \in \mathbb{R}$, all $\epsilon > \epsilon_0$ and all $T \geq T_0$,*

$$\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T} \prod_{j=1}^{k} G_T(\eta_j, t)\, dt = O_k\left(\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T} \prod_{j=1}^{k} L_T(M_k \eta_j, t)\, dt\right),$$

where the implied constant depends only upon k.

Proof. Note that there is an absolute constant C so that

$$\frac{1}{\epsilon}\mathbf{1}_{[a,a+\epsilon]}(x) \leq C V_{a,\epsilon}(x)$$

for

$$V_{a,\epsilon}(x) := \frac{1}{10\epsilon} V\Big(\frac{x-a}{10\epsilon}\Big)$$

where

$$V(x) := \Big(\tfrac{\sin \pi x}{\pi x}\Big)^2.$$

Because

$$\hat{V}_{a,\epsilon}(x)(\xi) = e^{i2\pi a\xi}(1 - 10\epsilon|x|)_+$$

is supported in $[-1/\epsilon_0, 1/\epsilon_0]$ for all $a \in \mathbb{R}$ and all $\epsilon > \epsilon_0$, we can apply Lemma 7.2 to bound the average in Lemma 7.3 from above. □

7.3. We now turn to the second upper bound outlined in section 6, for test functions with a narrowly supported Fourier transform. This is

Lemma 7.4. *For $\epsilon_0 > 0$, there exists a T_0 depending only on ϵ_0 so that for all $a \in \mathbb{R}$, all $\epsilon > \epsilon_0$, and all $g \in C_c^2(\mathbb{R})$ supported in $[-X, X]$ with $X \leq 1/k$, for all $T \geq T_0$*

$$\int_{\mathbb{R}} \frac{\mathbf{1}_{[a,a+\epsilon]}(t/T)}{\epsilon T}\big|\widetilde{G}_T(\,\hat{g}_\ell\,,t)\big|^k\,dt = O_k\Big(A^k\big(\tfrac{1}{\log^k T} + X^k\big)\Big),$$

where A is the maximum value of g.

To prove this bound we require another computational lemma that we will apply to Lemma 7.1.

Lemma 7.5. *For functions $g_1, ..., g_k$ each supported on the interval $[-X, X]$ and bounded in absolute value by a constant A, for $H \geq 1$ we have*

$$\frac{1}{H^k}\sum_{n_1^{\epsilon_1}n_2^{\epsilon_2}\cdots n_k^{\epsilon_k}=1}\prod_{\ell=1}^{k}\frac{\Lambda(n_\ell)}{\sqrt{n_\ell}}g\Big(\tfrac{\epsilon_\ell \log n_\ell}{H}\Big) = O_k(A^k X^k). \tag{43}$$

Remark: With control on the first and second derivatives of g_ℓ, a more exact evaluation can be made. See Lemma 12 of [**52**].

Proof of Lemma 7.5. We require from number theory only the Chebyshev estimate that

$$\sum_{p\leq x}\log p = O(x).$$

As the von Mangoldt function Λ is supported on prime powers p^λ, the sum in (43) is just

$$\frac{1}{H^k}\sum_{p_1^{\lambda_1\epsilon_1}\cdots p_k^{\lambda_k\epsilon_k}=1}\prod_{\ell=1}^{k}\frac{\log p_\ell}{p_\ell^{\lambda_\ell/2}}g\Big(\tfrac{\epsilon_\ell\lambda_\ell\log p_\ell}{H}\Big) \leq \frac{A^k}{H^k}\sum_{p_1^{\lambda_1\epsilon_1}\cdots p_k^{\lambda_k\epsilon_k}=1}\prod_{\ell=1}^{k}\frac{\log p_\ell}{p_\ell^{\lambda_\ell/2}}\mathbf{1}_{[0,x]}\Big(\tfrac{\lambda_\ell\log p_\ell}{H}\Big).$$

Here the sum ranges over all collections of k primes $\{p_1, ..., p_k\}$, k positive integers $\{\lambda_1, ..., \lambda_k\} \in \mathbb{N}_+^k$ and signs $\{\epsilon_1, ..., \epsilon_k\} \in \{-1, 1\}^k$ so that $p_1^{\lambda_1\epsilon_1}\cdots p_k^{\lambda_k\epsilon_k} = 1$. Owing to the weights $p^{\lambda/2}$, our main contribution comes from terms in which $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 1$. By unique factorization, $p^{\epsilon_1}\cdots p^{\epsilon_k} = 1$ only when each p_i is equal to some pair, p_j. As there are c_k

ways to form such pairs, where c_k is $(k-1)!!$ if k is even and 0 if k is odd,

$$\frac{A^k}{H^k}\sum_{p_1^{\epsilon_1}\cdots p_k^{\epsilon_k}=1}\prod_{\ell=1}^k \frac{\log p_\ell}{\sqrt{p_\ell}}\mathbf{1}_{[0,X]}\Big(\tfrac{\log p_\ell}{H}\Big) = A^k c_k\cdot\Big(\frac{1}{H^2}\sum_{\log p\leq XH}\frac{\log^2 p}{p}\Big)^{k/2}$$
$$= O_k(A^kX^k).$$

For the remaining terms in which one of $\lambda_1,...,\lambda_k$ is greater than 1, note that if $\lambda_1,\lambda_2,...,\lambda_k$ are each no less than 3,

$$\frac{A^k}{H^k}\sum_{\substack{p_1^{\lambda_1\epsilon_1}\cdots p_k^{\lambda_k\epsilon_k}=1\\ \lambda_1,...,\lambda_k\geq 3}}\prod_{\ell=1}^k \frac{\log p_\ell}{p_\ell^{\lambda_\ell/2}}\mathbf{1}_{[0,x]}\Big(\tfrac{\lambda_\ell\log p_\ell}{H}\Big) \leq \frac{A^k}{H^k}\Big(\sum_{\lambda\geq 3,p}\frac{\log p}{p^{\lambda/2}}\mathbf{1}_{[0,X]}\Big(\tfrac{\lambda_\ell\log p_\ell}{H}\Big)\Big)^k$$
$$= O_k\Big(\frac{A^k}{H^k}\Big).$$

But because the sum is 0 if $\frac{3\log 2}{H} > X$, this is $O_k(A^kX^k)$ all the same. Finally, if some λ_j is fixed to be equal to 2 – suppose without generality $j=1$ – then in our sum some p_i must equal p_1. If we with no loss of generality suppose the index i is 2, we have

$$\frac{A^k}{H^k}\sum_{p_1^{\lambda_1\epsilon_1}\cdots p_k^{\lambda_k\epsilon_k}=1}\prod_{\ell=1}^k \frac{\log p_\ell}{\sqrt{p_\ell}}\mathbf{1}_{[0,x]}\Big(\tfrac{\log p_\ell}{H}\Big)$$
$$\leq \frac{A^k}{H^k}\Big(\sum_p\sum_{\lambda_2\geq 1}\frac{\log^2 p}{p^{1+\lambda_2/2}}\Big)\sum_{p_3^{\lambda_3\epsilon_3}\cdots p_k^{\lambda_k\epsilon_k}=1}\prod_{\ell=3}^k \frac{\log p_\ell}{p_\ell^{\lambda_\ell/2}}\mathbf{1}_{[0,x]}\Big(\tfrac{\lambda_\ell\log p_\ell}{H}\Big)$$
$$= O\Bigg(\frac{A^2}{H^2}\frac{A^{k-2}}{H^{k-2}}\sum_{p_3^{\lambda_3\epsilon_3}\cdots p_k^{\lambda_k\epsilon_k}=1}\prod_{\ell=3}^k \frac{\log p_\ell}{p_\ell^{\lambda_\ell/2}}\mathbf{1}_{[0,x]}\Big(\tfrac{\lambda_\ell\log p_\ell}{H}\Big)\Bigg).$$

An inductive argument shows this is $O_k(A^k/H^k)$, and again, for the sum to be nonzero we must have $1/H \lesssim X$. Since there are only k such cases that some λ_j may be fixed to be 2, we have shown that the sum (43) is $O_k(A^kX^k)$. □

From Lemmas 7.1 and 7.5,

Corollary 7.6. *Suppose we are given non-negative integrable σ of mass 1 such that $\hat{\sigma}$ has compact support, and suppose $g_1,...,g_k$ are in $C_c^2(\mathbb{R})$ and each supported in a region $[-X,X]$ with $X\leq 1/k$ and each bounded in absolute value by a constant A. Then there exists a T_0 depending only on the region in which $\hat{\sigma}$ is supported so that for $T\geq T_0$,*

$$\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\prod_{\ell=1}^k \widetilde{G}_T(\,\hat{g}_\ell\,,t)\,dt = O_k\Big(A^k\big(\tfrac{1}{\log^k T}+X^k\big)\Big).$$

With a little more work,

Corollary 7.7. *For σ as above in Corollary 7.6 and $g\in C_c^2(\mathbb{R})$ supported in $[-X,X]$ with $X\leq 1/k$ and bounded in absolute value by a constant A, there exists T_0 depending only on the region in which $\hat{\sigma}$ is supported so that for $T\geq T_0$,*

$$\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\big|\widetilde{G}_T(\,\hat{g}_\ell\,,t)\big|^k\,dt = O_k\Big(A^k\big(\tfrac{1}{\log^k T}+X^k\big)\Big).$$

PROOF. By Cauchy-Schwarz,

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \left|\widetilde{G}_T(\,\hat{g}_\ell\,,t)\right|^k dt \leq \sqrt{\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \left(\widetilde{G}_T(\,\hat{g}_\ell\,,t)\right)^k \left(\widetilde{G}_T(\,\overline{\hat{g}_\ell}\,,t)\right)^k dt}.$$

But

$$\overline{\hat{g}(\xi)} = \int_{\mathbb{R}} e(-x\xi)\overline{g}(-x)\,dx$$

and $\overline{g}(-x)$ is also bounded in absolute value by A and supported in $[-X, X]$, so the corollary follows from Corollary 7.6. □

Remark: Corollary 7.7 says that the kth moment of $\left|\widetilde{G}_T(\,\hat{g}_\ell\,,t)\right|$ is small. This cannot be immediately inferred from Corollary 7.6 because if $\hat{g}_\ell$ is signed, there may be cancellation in the integral taken there.

Lemma 7.4 then follows exactly in the same way as Lemma 7.3, by majorizing $\frac{1}{\epsilon}\mathbf{1}_{[a,a+\epsilon]}$ by $V_{a,\epsilon}$, exploiting the compactly supported Fourier tranform of the latter.

8. Upper bounds for counts of eigenvalues

In order to produce similar bounds for counts of eigenvalues, we need an analogue of Lemma 7.1. This is furnished by a result of Diaconis and Shahshahani [**13**]:

THEOREM 8.1 (Diaconis-Shahshahani). *Let $\mathcal{U}(n)$ be the set of $n\times n$ unitary matrices endowed with Haar measure. Consider $a = (a_1, ..., a_k)$ and $b = (b_1, ..., b_k)$ with $a_1, a_2, ..., b_1, b_2, ... \in \{0, 1, ...\}$.If $\sum_{j=1}^k ja_j \neq \sum_{j=1}^k jb_j$,*

$$\int_{\mathcal{U}(n)} \prod_{j=1}^k \mathrm{Tr}(g^j)^{a_j}\overline{\mathrm{Tr}(g^j)^{b_j}}\,dg = 0. \tag{44}$$

Furthermore, in the case that

$$\max\left(\sum_{j=1}^k ja_j, \sum_{j=1}^k jb_j\right) \leq n$$

we have

$$\int_{\mathcal{U}(n)} \prod_{j=1}^k \mathrm{Tr}(g^j)^{a_j}\overline{\mathrm{Tr}(g^j)^{b_j}}\,dg = \delta_{ab}\prod_{j=1}^k j^{a_j}a_j!. \tag{45}$$

A simple manipulation in enumerative combinatorics allows us to rephrase (45) as the statement that for integers $j_1, ..., j_k$ such that $|j_1| + \cdots + |j_k| \leq 2N$,

$$\int_{\mathcal{U}(N)} \prod_{\ell=1}^k \mathrm{Tr}(u^{j_\ell})\,du = \sum \prod_\lambda |j_{\mu_\lambda}|\delta_{j_{\mu_\lambda}=-j_{\nu_\lambda}},$$

where the sum is over all partitions of $[k] = \{1, ..., k\}$ into disjoint pairs $\{\mu_\lambda, \nu_\lambda\}$ and $\delta_{j_{\mu_\lambda}=-j_{\nu_\lambda}}$ is 1 or 0 depending upon whether $j_{\mu_\lambda} = -j_{\nu_\lambda}$. For instance, $\{1, 2, 3, 4\}$ can be partitioned into the disjoint pairs $\{\{1, 2\}; \{3, 4\}\}$, $\{\{1, 3\}; \{2, 3\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$, and we have

$$\int_{\mathcal{U}(N)} \mathrm{Tr}(u^{j_1})\mathrm{Tr}(u^{j_2})\mathrm{Tr}(u^{j_3})\mathrm{Tr}(u^{j_4})\, du = |j_1|\delta_{j_1=-j_2}|j_3|\delta_{j_3=-j_4} + |j_1|\delta_{j_1=-j_3}|j_2|\delta_{j_2=-j_4} + |j_1|\delta_{j_1=-j_4}|j_2|\delta_{j_2=-j_3},$$

when $|j_1| + |j_2| + |j_3| + |j_4| \leq 2N$.

For the point processes $\mathcal{S}'_N$, by using Poisson summation as in identity (30),

Corollary 8.2. *For $g_1, ..., g_k \in C_c^2(\mathbb{R})$ satisfying* supp $g_\ell \subset [-\delta_\ell, \delta_\ell]$ *with $\delta_1 + \cdots + \delta_k \leq 2$,*

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N} \prod_{\ell=1}^k \Big(\sum_i \hat{g}_\ell(x_i) - \int_{-\infty}^{\infty} \hat{g}_\ell(\alpha)\, d\alpha \Big) = \sum \prod_\lambda \Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{N}\frac{|j|}{N} g_{\mu_\lambda}\Big(\tfrac{j}{N}\Big) g_{\nu_\lambda}\Big(\tfrac{-j}{N}\Big)\Big),$$

where the first sum is, as above, over all partitions of $[k]$ into disjoint paris $\{\mu_\lambda, \nu_\lambda\}$.

8.1. With proofs proceeding exactly as in section 4, we obtain an analogue of Lemma 7.4,

Corollary 8.3. *For $g_1, ..., g_k \in C_c^2(\mathbb{R})$ each supported in a region $[-X, X]$ with $X \leq 1/k$ and each bounded in absolute value by a constant A,*

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N} \prod_{\ell=1}^k \Big(\sum_i \hat{g}_\ell(x_i) - \int_{-\infty}^{\infty} \hat{g}_\ell(\alpha)\, d\alpha \Big) = O_k(A^k X^k).$$

Corollary 8.4. *For $g \in C_c^2(\mathbb{R})$ supported in $[-X, X]$ with $X \leq 1/k$, and with maximum value A,*

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N} \Big| \sum_i \hat{g}(x_i) - \int_{-\infty}^{\infty} \hat{g}(\alpha)\, d\alpha \Big|^k = O_k(A^k X^k).$$

8.2. In the same way, we can produce an analogue of Fujii's bound:

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N} \Big| \sum_j \eta(x_j) \Big|^k \lesssim_k \Big| \int_{\mathbb{R}} M_k \eta(\alpha)\, d\alpha \Big|^k.$$

For our purposes this is rendered redundant by our ability to explicitly calculate the correlation functions of $\mathcal{S}'_N$, and in particular by knowing Proposition 5.8 – that $\mathcal{S}'_N \to S$.

9. A Tauberian interchange of averages

9.1. Recall that for a weight σ, GUE(σ) is an abbreviation for the proposition that the processes $Z_T(\sigma)$ tend in correlation to the sine-kernel determinantal process $\mathcal{S}$. In this section we show that for many σ, the proposition GUE(σ) is equivalent to GUE($\mathbf{1}_{[1,2]}$), that is to say the GUE Conjecture proper.

We use the abbreviation

$$d\lambda_k(t) := \log^k(|t| + 2)\, dt.$$

Theorem 9.1. *Let $\sigma_1(t)$ and $\sigma_2(t)$ be non-negative piecewise continuous functions on $\mathbb{R}$ of mass 1 both dominated by a function $\varsigma(t)$ which decreases radially and is an element of $L^1(\mathbb{R}, d\lambda_k)$ for all $k \geq 1$. If for $f_1(x) = e^x \sigma_1(e^x)$ we have $\hat{f}_1(\xi) \neq 0$ for all ξ, then*

$$\mathrm{GUE}(\sigma_1) \Rightarrow \mathrm{GUE}(\sigma_2).$$

Our proof makes use of the first upper bound in section 7, the fact that the density of zeros grows sub-polynomially, and finally Weiner's Tauberian theorem to relate a specific σ to other weights.

9.2. We first develop an upper bound in terms of the weight σ. As a corollary of Lemma 7.3, making a change of variables $\tau = t/T$ and on the right, recalling the definition (32) of L_T, making the change of variables $x = \frac{\log T}{2\pi}(\xi - T\tau)$,

COROLLARY 9.2. *For $\epsilon > 0$ there exists T_0 such that for $T \geq T_0$,*

$$\int \mathbf{1}_{[a,a+\epsilon]}(\tau) \prod_{j=1}^{k} G_T(\eta_j, T\tau)\, d\tau$$

$$\lesssim_k \prod_{j=1}^{k} \|M_k \eta_j\|_{L^1(d\lambda_1)} \bigg(\int \mathbf{1}_{[a,a+\epsilon]}(\tau)\, d\tau + \int \mathbf{1}_{[a,a+\epsilon]}(\tau) \frac{\log^k(|\tau|+2)}{\log^k T}\, d\tau \bigg).$$

for all $a \in \mathbb{R}$ and functions $\eta_1, ..., \eta_k$.

Remark: The importance of this bound is that its implied constant (and T_0) is independent of a and test functions η.

From this,

COROLLARY 9.3. *For σ_1 piecewise continuous and dominated by a function ς as in Theorem 9.1, for $T \geq T_0$*

$$\int_{\mathbb{R}} \sigma_1(\tau) \prod_{j=1}^{k} G_T(\eta_j, T\tau)\, d\tau \lesssim_k \prod_{j=1}^{k} \|M_k \eta_j\|_{L^1(d\lambda_1)} \Big(\|\sigma_1\|_{L^1(d\tau)} + \tfrac{1}{\log^k T} \|\sigma_1\|_{L^1(d\lambda_k)} \Big)$$

where T_0 depends only on ς and σ_1 and the implied constant only on k.

PROOF OF COROLLARY 9.3. Fix k. Let δ be an arbitrary positive number, and choose K so that

$$\int_{|t|>K} \varsigma(\tau) \log^k(|\tau|+2)\, d\tau < \delta.$$

Likewise, choose ϵ positive but less than 1 so that

$$\int_{|\tau|<K+1} \big(M_\epsilon \sigma_1(\tau) - \sigma_1(\tau)\big) \log^K(|\tau|+2)\, d\tau < \delta.$$

We have that

$$\int \sigma_1(\tau) \prod_{j=1}^{k} G_T(\eta_j, T\tau)\, d\tau \lesssim \int M_\epsilon \sigma_1(\tau) \prod_{j=1}^{k} G_T(\eta_j, T\tau)\, d\tau$$

$$\lesssim_k \prod_{j=1}^{k} \|M_k \eta_j\|_{L^1(d\lambda_1)} \bigg(\int M_\epsilon \sigma_1(\tau)\, d\tau + \int M_\epsilon \sigma_1(\tau) \tfrac{\log^k(|\tau|+2)}{\log^k T}\, d\tau \bigg)$$

for $T \geq T_0$ depending only upon ϵ.

Because ς decays away from the origin and dominates σ_1,

$$\int_{|\tau|>K+1} M_\epsilon \sigma_1(\tau) \log^k(|\tau|+2)\, d\tau < \delta,$$

and so for $T \geq T_0$,

$$\int \sigma_1(\tau) \prod_{j=1}^k G_T(\eta_j, T\tau)\, d\tau \lesssim_k \prod_{j=1}^k \|M_k \eta_j\|_{L^1(d\lambda_1)} \Big(\int_{|\tau<K+1} M_\epsilon \sigma_1(\tau) \Big(1 + \tfrac{\log^k(|\tau|+2)}{\log^k T}\Big)\, d\tau + \delta \cdot \Big(1 + \tfrac{1}{\log^k T}\Big)\Big)$$

$$\lesssim_k \prod_{j=1}^k \|M_k \eta_j\|_{L^1(d\lambda_1)} \Big(\int \sigma_1(\tau) \Big(1 + \tfrac{\log^k(|\tau|+2)}{\log^k T}\Big)\, d\tau + 2\delta \cdot \Big(1 + \tfrac{1}{\log^k T}\Big)\Big).$$

As δ was arbitrary, we can let it be smaller for instance than $\|\sigma_1\|_{L^1(dt)}$ and obtain,

$$\int \sigma_1(\tau) \prod_{j=1}^k G_T(\eta_j, T\tau)\, d\tau \lesssim_k \prod_{j=1}^k \|M_k \eta\|_{L^1(d\lambda_1)} \Big(\|\sigma_1\|_{L^1(dt)} + \frac{1}{\log^k T} \|\sigma_1\|_{L^1(d\lambda_k)} \Big)$$

for sufficiently large T depending only upon ς and σ_1. □

9.3. Before proceeding to a proof of Theorem 9.1, we embark on a small digression. Corollary 9.3 yields a quick way to see that there is nothing special about using $C_c(\mathbb{R}^k)$ functions to test whether $Z_T(\sigma) \to \mathcal{S}$ in correlation.

Proposition 9.4. *For each $k \geq 1$, let $\mathcal{A}_k$ be a collection of functions $\eta\colon \mathbb{R}^k \to \mathbb{R}$ such that*

(i) For any $\eta \in \mathcal{A}_k$, η decays in each variable at a $3/2$-power rate; that is, there is a constant A_η so that

$$|\eta(x_1, ..., x_k)| \leq \frac{A_\eta}{(1+|x_1|^{3/2}) \cdots (1+|x_k|^{3/2})},$$

and more

(ii) For any $\rho \in C_c(\mathbb{R})$ any any $\epsilon > 0$, there exists $\eta \in \mathcal{A}_k$ so that for all $x \in \mathbb{R}^k$,

$$|\rho(x) - \eta(x)| \leq \frac{\epsilon}{(1+|x_1|^{3/2}) \cdots (1+|x_k|^{3/2})}.$$

Then for any $\sigma_1\colon \mathbb{R} \to \mathbb{R}_+$ positive, piecewise continuous, and of mass 1, and dominated by a function ς as in Theorem 9.3, GUE(σ_1) *is equivalent to the statement that for all $k \geq 1$ and all $\eta \in \mathcal{A}_k$,*

$$\mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(\xi_{j_1}, ..., \xi_{j_k}) = \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(x_{j_1}, ..., x_{j_k}) + o(1). \tag{46}$$

Remark: If not for the fact that the collections $\mathcal{A}_k$ may contain η which are not compactly supported, this proposition would be standard. The $3/2$ power decay in (i) and (ii) is chosen for convenience rather than canonically. Some decay in the tails of functions η is necessary for the proposition to be true, and for technical reasons later on to have a proposition with for η whose tails decay more slowly than quadratically will be important.

Proof of Proposition 9.4. Recall that GUE(σ_1) is equivalent to the statement that for all $k \geq 1$ and all $\rho \in C_c(\mathbb{R}^k)$,

$$\mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \rho(\xi_{j_1}, .., \xi_{j_k}) = \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \rho(x_{j_1}, .., x_{j_k}) + o(1).$$

By inductively including lower correlations, we see that this is equivalent to the statement that for all $k \geq 1$ and $\rho \in C_c(\mathbb{R}^k)$,

$$\mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{j_1,\dots,j_k} \rho(\xi_{j_1},..,\xi_{j_k}) = \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{j_1,\dots,j_k} \rho(x_{j_1},..,x_{j_k}) + o(1).$$

The sums here are over indices which needn't be distinct. By applying Corollary 9.3 for sufficiently large T, for any η,

$$\Big| \mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{j_1,\dots,j_k} \eta(\xi_{j_1},...,\xi_{j_k}) \Big| \lesssim_{k,\sigma_1} \int_{\mathbb{R}^k} M_k' \eta(x_1,...,x_k)\, d\lambda_1(x_1)\cdots d\lambda(x_k),$$

where

$$M_k'\eta(x_1,...,x_k) = \sum_{\nu\in\mathbb{Z}^k} \Big(\sup_{I_k'(\nu)} |\eta| \Big) \mathbf{1}_{I_k'(\nu)}(x),$$

where $I_k'(\nu)$ abbreviates the k-dimensional cube $k\nu + [-k/2, k/2)^k$.

Note that for any $\epsilon > 0$, any $\eta\colon \mathbb{R}^k \to \mathbb{R}^k$ which decays in each variable in the sense of condition (i) can be approximated by $\rho \in C_c(\mathbb{R}^k)$ so that both

$$\Big| \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{j_1,\dots,j_k} \big(\eta(x_{j_1},...,x_{j_k}) - \rho(x_{j_1},...,x_{j_k})\big) \Big| < \epsilon,$$

and

$$\int_{\mathbb{R}^k} M_k'(\eta-\rho)\, d\lambda(x_1)\cdots d\lambda(x_k) < \epsilon.$$

It therefore follows that for continuous $\eta\colon \mathbb{R}^k \to \mathbb{R}$ decaying in each variable as in (i), GUE(σ_1) implies

$$\varlimsup_{T\to\infty} \Big| \mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{j_1,\dots,j_k} \eta(\xi_{j_1},...,\xi_{j_k}) - \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{j_1,\dots,j_k} \eta(\xi_{j_1},...,\xi_{j_k}) \Big| < 2\epsilon.$$

Because ϵ is arbitrary, this shows that GUE(σ_1) implies (46) for any $\eta \in \mathcal{A}_k$.

In the opposite direction, suppose that for all $k \geq 1$ and any $\eta \in \mathcal{A}_k$, (46) holds. Let ρ be an arbitrary element of $C_c(\mathbb{R}^k)$. For any $\epsilon > 0$, there exists an $\eta \in \mathcal{A}_k$ so that for all $x \in \mathbb{R}^k$,

$$|\eta(x) - \rho(x)| < \frac{\epsilon}{(1+|x_1|^{3/2})\cdots(1+|x_k|^{3/2})}.$$

Thus it follows as before that

$$\begin{aligned} &\Big| \mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \rho(\xi_{j_1},...,\xi_{j_k}) - \mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \eta(\xi_{j_1},...,\xi_{j_k}) \Big| \\ &\leq \mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{j_1,\dots,j_k} \big|\rho(\xi_{j_1},...,\xi_{j_k}) - \eta(\xi_{j_1},...,\xi_{j_k})\big| \\ &\lesssim_{k,\sigma_1} \epsilon \end{aligned}$$

and

$$\Big| \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \rho(x_{j_1},...,x_{j_k}) - \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \eta(x_{j_1},...,x_{j_k}) \Big| \lesssim_k \epsilon.$$

As ϵ was arbitrary it follows that

$$\mathop{\mathbf{E}}_{Z_T(\sigma_1)} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \rho(\xi_{j_1},...,\xi_{j_k}) = \mathop{\mathbf{E}}_{\mathcal{S}} \sum_{\substack{j_1,\dots,j_k \\ \text{distinct}}} \rho(x_{j_1},...,x_{j_k}) + o(1).$$

Because ρ was arbitrary, this is just GUE(σ_1). □

9.4. We return to the proof of Theorem 9.1. Recall Weiner's Tauberian Theorem:

THEOREM 9.5 (Weiner). *For $f_1, f_2 \in L^1(\mathbb{R}, dt)$ with $\hat{f}_1(\xi) \neq 0$, for any $\epsilon > 0$ there exist constants $w_1, ..., w_n$ and $a_1, ..., a_n$ so that*

$$\|f_2(t) - \sum a_i f_1(t - w_i)\|_{L^1(dt)} < \epsilon.$$

That is, $\mathrm{span}_{w\in\mathbb{R}}\{f(t-w)\}$ is dense in $L^1(\mathbb{R}, dt)$. See for instance [**39**] for a proof.

With this we can proceed to a

PROOF OF THEOREM 9.1. Choose $\epsilon > 0$. Weiner's Tauberian Theorem implies that there exist positive $h_1, ..., h_n$ and (possibly negative) $a_1, ..., a_n$ so that $a_1 + \cdots + a_n = 1$ and

$$\|\sigma_1(\tau) - \sum a_i h_i^{-1} \sigma_1(\tau/h_i)\|_{L^1(dt)} < \epsilon.$$

It is because σ_2 and σ_1 are both of mass 1 that we can choose $a_1, ..., a_n$ so that $a_1+\cdots+a_n = 1$. Because linear combinations of separable and continuously differentiable functions are dense in $C_c(\mathbb{R}^k)$, an expansion into lower order correlations shows that for σ either of σ_1 or σ_2, GUE(σ) is equivalent to the statement that for all k and continuously differentiable and compactly supported η_1, ..., η_k,

$$\lim_{T\to\infty} \mathop{\mathbf{E}}_{Z_T(\sigma)} \prod_{\ell=1}^{k} \sum_i \eta_\ell(\xi_i) = \prod_{\ell=1}^{k} \sum_i \eta_\ell(\xi_i).$$

Because any continuously differentiable η can be written as the difference of two radially non-increasing functions, e.g. for $x > 0$,

$$\eta(x) = \Big(\int_x^\infty \Big(\frac{d\eta}{dx}\Big)_+ dx\Big) - \Big(\int_x^\infty -\Big(\frac{d\eta}{dx}\Big)_- dx\Big),$$

GUE(σ) is equivalent to the statement that

$$\begin{aligned}\int \sigma(\tau) \prod_{\ell=1}^{k} G_T(\eta_\ell, T\tau)\, d\tau &= \mathop{\mathbf{E}}_{Z_T(\sigma)} \prod_{\ell=1}^{k} \sum_i \eta_j(\xi_i) \\ &= \mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^{k} \sum_i \eta_j(\xi_i) + o(1)\end{aligned}$$

for any collection $\eta_1, ..., \eta_j$ of radially non-increasing functions, continuous and compactly supported.

We make use of a monotonicity argument to show that on the hypothesis of Theorem 9.1 for any $h > 0$,

$$h^{-1} \int \sigma_1\Big(\frac{\tau}{h}\Big) \prod_{\ell=1}^{k} G_T(\eta_\ell, T\tau)\, d\tau = \mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^{k} \sum_i \eta_\ell(\xi_i) + o(1). \tag{47}$$

Clearly this is true for $h = 1$. For other h, the left hand side of (47) is equal to

$$\int \sigma_1(\tau) \prod_{\ell=1}^{k} G_T(\eta_\ell, Th\tau)\, d\tau.$$

If we define $\eta[\rho](x) := \eta(\rho^{-1}x)$, then for $\rho_1 < \rho_2$ (as long as η is non-increasing radially) $\eta[\rho_1] \leq \eta[\rho_2]$ pointwise. Also note

$$\begin{aligned} G_T(\eta_\ell, Th\tau) &= \sum_\gamma \eta_\ell\Big(\tfrac{\log T}{2\pi}(\gamma - Th\tau)\Big) \\ &= G_{Th}\Big(\eta_\ell\big[1+\tfrac{\log h}{\log T}\big], Th\tau\Big). \end{aligned}$$

We consider first the case that $h < 1$. In this case, for $T > T'$ (because the quantity $1 + \frac{\log h}{\log T}$ decreases as T increases),

$$\begin{aligned} \int \sigma_1(\tau) \prod_{\ell=1}^k G_T(\eta_\ell, Th\tau)\, d\tau &\leq \int \sigma_1(\tau) \prod_{\ell=1}^k G_{Th}\Big(\eta_\ell\big[1+\tfrac{\log h}{\log T'}\big], Th\tau\Big)\, d\tau \\ &= \mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^k \Big(\sum_i \eta_\ell\big[1+\tfrac{\log h}{\log T'}\big](\xi_i)\Big) + o(1) \end{aligned} \tag{48}$$

For the same reason,

$$\begin{aligned} \int \sigma_1(\tau) \prod_{\ell=1}^k G_T(\eta_\ell, Th\tau)\, d\tau &\geq \int \sigma_1(\tau) \prod_{\ell=1}^k G_{Th}(\eta_\ell, Th\tau)\, d\tau \\ &= \mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^k \Big(\sum_i \eta_\ell(\xi_i)\Big) + o(1). \end{aligned} \tag{49}$$

As $T \to \infty$, we may choose T' arbitrarily large, and because the resulting limiting expression in (48) is continuous in $\frac{\log h}{\log T'}$, we have (47) as claimed.

In the case that $h < 1$, we may use the same argument, with the inequalities in both (48) and (49) reversed.

To complete the proof, note that by Corollary 9.3

$$\begin{aligned} &\overline{\lim}_{T\to\infty} \Big| \int_{-\infty}^{\infty} \big(\sigma_2(\tau) - \sum a_i h_i \sigma_1(\tau/h_i)\big) \prod_{\ell=1}^k G_T(\eta_\ell, T\tau)\, d\tau \Big| \\ &\lesssim_{\eta,k} \overline{\lim}_{T\to\infty} \Big(\Big\|\sigma_2(\tau) - \sum a_i h_i \sigma_1(\tau/h_i)\Big\|_{L^1(dt)} + \frac{1}{\log^k T} \Big\|\sigma_2(\tau) - \sum a_i h_i \sigma_1(\tau/h_i)\Big\|_{L^1(d\lambda_k(t))} \Big) \\ &< \epsilon. \end{aligned}$$

Because ϵ was arbitrary, (47) and the fact that $a_1 + \cdots + a_n = 1$ yield that

$$\begin{aligned} \mathop{\mathbf{E}}_{Z_T(\sigma_2)} \prod_{\ell=1}^k \sum_i \eta_\ell(\xi_i) &= \int \sigma_2(\tau) \prod_{\ell=1}^k G_T(\eta_\ell, T\tau)\, d\tau \\ &= \mathop{\mathbf{E}}_{\mathcal{S}} \prod_{\ell=1}^k \sum_i \eta_\ell(x_i) + o(1), \end{aligned}$$

as claimed. □

Remark: Note that instead of the monotonicity argument we have used above, which relies critically on the positivity of counts of zeros, one could also proceed alternatively without this information, by using Corollary 7.3 to show that

$$\int \sigma_t(\tau) \prod_{\ell=1}^{k} G_{Th}\Big(\eta_\ell[1+\log h/\log T], Th\tau\Big)\, d\tau - \int \sigma_1(\tau) \prod_{\ell=1}^{k} G_{Th}\Big(\eta_\ell, Th\tau\Big)\, d\tau \to 0.$$

This method of proof does have an advantage: in this way we could prove the theorem for weights σ_1 and σ_2 that are not non-negative. We will not need to consider such weights in what follows however.

9.5. We note two masses σ for which GUE(σ) reproduces itself to other masses.

COROLLARY 9.6. *The* GUE *Conjecture (*GUE($\mathbf{1}_{[1,2]}$)*, that is) implies* GUE(σ_2) *for any* σ_2 *which is piecewise continuous, in* $L^1(d\lambda_k)$ *for all* k*, and dominated by a decreasing function.*

PROOF. It is apparent that $\sigma_1 := \mathbf{1}_{[1,2]}$ is itself non-negative, in $L^1(d\lambda_k)$ for all k, and non-increasing radially. In addition, the function $f_1(t) := e^t \mathbf{1}_{[1,2]}(e^t)$ satisfies

$$\hat{f}_1(\xi) = \frac{2^{1-i2\pi\xi} - 1}{1 - i2\pi\xi} \neq 0,$$

for all ξ. □

Likewise,

COROLLARY 9.7. *For*

$$\sigma_1(t) := \frac{1}{2\pi}\Big(\frac{\sin t/2}{t/2}\Big)^2, \tag{50}$$

GUE(σ_1) *implies* GUE(σ_2) *for any* σ_2 *which is piecewise continuous, in* $L^1(d\lambda_k)$ *for all* k*, and dominated by a decreasing function.*

PROOF. Again it is apparent that σ_1 is non-negative and may be dominated by a function that is in $L^1(d\lambda_k)$ for all k and non-increasing radially. If $f_1(t) := e^t\sigma_1(e^t)$, then

$$\hat{f}_1(\xi) = \frac{\Gamma(-i2\pi\xi)\sin(-i\pi^2\xi)}{\pi(1-i2\pi\xi)} \neq 0$$

for all ξ. □

COROLLARY 9.8. *The GUE Conjecture is equivalent to GUE*(σ_1) *where* σ_1 *is defined in* (50)*.*

For us the significance of this particular σ_1 is that

$$\hat{\sigma}_1\big(\tfrac{x}{2\pi}\big) = (1-|x|)_+.$$

10. Approximating a principal value integral

10.1. We have come to the point to introduce the cutoff $f|_\epsilon$ of functions f mentioned in the outline in section 6. Recall (27) and (28), the definition of the bump function α and rescaled bump function α_ϵ of width 2ϵ. (Earlier our interest was a rescaling with large width, in the context of the present chapter, we rescale to small width.) The reader should check that $\alpha(0) = 1$ and $\alpha'(0) = \alpha''(0) = 0$. Using α_ϵ, we define

$$\omega_\epsilon(x) := 1 - \alpha_\epsilon(x)$$

$$\Omega_\epsilon(x) := \omega_\epsilon(x)\mathbf{1}_{\mathbb{R}_+}(x).$$

It is easy to verify that $\Omega_\epsilon \in C^2(\mathbb{R})$.

We define the cutoff function $f|_\epsilon$ for $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f|_\epsilon(x) := f(x)\Omega_\epsilon(x).$$

For small ϵ this approximates $f \cdot \mathbf{1}_{\mathbb{R}_+}$. Further, for $b > a > 0$ we define

$$f|_a^b(x) := f|_a(x) - f|_b(x),$$

which is supported on the interval $[0, b]$ and morally acts as a restriction of f to the interval $[a, b]$.

10.2. The purpose of this section is to show that

LEMMA 10.1. *For admissible g (see definition 2.3), and non-negative and integrable σ such that $\hat{\sigma}$ is compactly supported, there exists T_0 depending only on the region in which $\hat{\sigma}$ is supported so that for all $T > T_0$ and all $\epsilon > 0$,*

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_\epsilon)\hat{\ }, t\big)\big|^k \, dt \lesssim_k \|g\|_{L^1(\mathbb{R})}^k + \|g'\|_{L^1(\mathbb{R})}^k + \|g''\|_{L^1(\mathbb{R})}^k$$

for $k \geq 1$.

This lemma may be at first surprising in the same way as the upper bound Lemma 7.4. In fact, it is true for much the same reason as Lemma 7.4. A partial explanation for the bound is that while $(g\mathbf{1}_{\mathbb{R}_+})\hat{\ }$ is not integrable for g smooth and $g(0) \neq 0$, for such g the principal value integral

$$\lim_{R\to\infty} \int_{-R}^{R} (g \cdot \mathbf{1}_{\mathbb{R}_+})\hat{\ }(\xi) \, d\xi$$

has the limit

$$= \tfrac{1}{2} g(0),$$

owing to the oscillatory nature of $g\mathbf{1}_{\mathbb{R}_+}$. For small ϵ, $g|_\epsilon$ resembles $g\mathbf{1}_{\mathbb{R}_+}$ and so in particular $\|g|_\epsilon\|_{L^1}$ will grow without bound. But at the same time $(g|_\epsilon)\hat{\ }$ will capture the same oscillation as $(g\mathbf{1}_{\mathbb{R}_+})\hat{\ }$ and (much as in Lemma 7.4), this substantially reduces the size of $\widetilde{G}_T((g|_\epsilon)\hat{\ }, t)$.

10.3. In proving Lemma 10.1, it will be useful to have in mind some standard explicit bounds on the decay of $\hat{g}$ for $g \in C_c^2(\mathbb{R})$. Note that

$$\hat{g}(\xi) = -\frac{1}{4\pi^2\xi^2} \int_{\mathbb{R}} g''(x) e(-x\xi) \, dx$$

and because we have for all ξ (in particular for ξ close to the origin),

$$|\hat{g}(\xi)| \leq \|g\|_{L^1(\mathbb{R})},$$

we have the estimate

$$\hat{g}(\xi) = O\Big(\frac{\|g\|_1 + \|g''\|_1}{\xi^2 + 1}\Big). \tag{51}$$

With this in mind,

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} L_T(\,|\hat{g}|\,, t)^k \, dt = O_k\big((\|g\|_{L^1} + \|g''\|_{L^1})^k\big). \tag{52}$$

and so a trivial consequence then of Lemma 7.2 is

Lemma 10.2. *For σ non-negative and integrable such that $\hat{\sigma}$ is compactly supported, there exists a T_0 depending only on the region in which $\hat{\sigma}$ is supported, so that for all $T \geq T_0$,*

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} G_T(\,|\hat{g}|\,, t)^k\, dt = O_k\big((\|g\|_{L^1} + \|g''\|_{L^1})^k\big).$$

10.4. From this, it is a short path to Lemma 10.1.

Proof of Lemma 10.1. From Minkowski's inequality,

$$\begin{aligned}\Big(\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_\epsilon)\hat{}\,, t\big)\big|^k\, dt\Big)^{1/k} \leq & \Big(\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_\epsilon^{1/k})\hat{}\,, t\big)\big|^k\, dt\Big)^{1/k} \\ & + \Big(\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_{1/k})\hat{}\,, t\big)\big|^k\, dt\Big)^{1/k}.\end{aligned} \tag{53}$$

From Lemma 7.4, there is T_0 so that for $T \geq T_0$,

$$\begin{aligned}\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_\epsilon^{1/k})\hat{}\,, t\big)\big|^k\, dt &\lesssim \|g\|_\infty^k \Big(\frac{1}{\log^k T} + \Big(\frac{1}{k}\Big)^k\Big) \\ &\lesssim \|g\|_\infty^k.\end{aligned} \tag{54}$$

On the other hand, applying equation (52) and its consequence, Lemma 10.2,

$$\begin{aligned}\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \big|\widetilde{G}_T\big((g|_{1/k})\hat{}\,, t\big)\big|^k\, dt &\lesssim_k \int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \Big(\big|G_T\big((g|_{1/k})\hat{}\,, t\big)\big|^k + \big|L_T\big((g_{1/k}\hat{}\,; t\big)\big|^k\, dt \\ &\lesssim_k \|\Omega_{1/k} g\|_{L^1}^k + \|(\Omega_{1/k} g)''\|_{L^1}^k \\ &\lesssim_k \|g\|_{L^1} + \|g'\|_{L^1} + \|g''\|_{L^1}\end{aligned} \tag{55}$$

as $\Omega_{1/k}, \Omega'_{1/k}$, and $\Omega''_{1/k}$ are all bounded. (Here we have repeatedly used the inequality $(a+b)^k \lesssim_k a^k + b^k$.)

Substituting (54) and (55) into (53) gives the lemma. □

11. Zeros and arithmetic

11.1. From the Tauberian result, Corollary 9.8, the GUE Conjecture is equivalent to the claim GUE(σ_1), for σ_1 defined in (50). In this section we prove Theorem 2.4. Our proof is broken into two parts; we first show that the GUE Conjecture implies the identity (12) for admissible functions, and in a separate second proof we demonstrate the converse.

11.2.

Proof of Theorem 2.4: The GUE Conjecture implies (12). We begin by establishing that for fixed admissible f, there exists some positive ϵ_T (depending on T) so that

$$\widetilde{G}_T\big[(f|_{\epsilon_T})\hat{}\,, t\big] = \frac{-1}{\log T} \int_{-\infty}^{\infty} f\Big(\tfrac{x}{\log T}\Big) e^{ixt}\, dz(x) + O_f\Big(\tfrac{1}{\log T}\Big), \tag{56}$$

and (letting $\check{f}(x) = \int f(\xi) e(x\xi)\, d\xi$ denote the inverse Fourier transform),

$$\widetilde{G}_T\big[(f|_{\epsilon_T})\check{}\,, t\big] = \frac{-1}{\log T} \int_{-\infty}^{\infty} f\Big(\tfrac{x}{\log T}\Big) e^{-ixt}\, dz(x) + O_f\Big(\tfrac{1}{\log T}\Big), \tag{57}$$

For, for admissible f, there is some $\alpha < 1/2$ such that

$$\frac{1}{\log T}\int_{-\infty}^{0} f\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x) = O_f\bigg(\frac{1}{\log T}\int_{-\infty}^{0} e^{x(1/2-\alpha)}\,dx\bigg) = O_f\Big(\tfrac{1}{\log T}\Big).$$

and by continuity there exists some $\epsilon_T > 0$ so that

$$\frac{1}{\log T}\int_0^{\infty}(f - f|_{\epsilon_T})\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x) \leq \frac{1}{\log T}$$

as $(f - f|_\epsilon)(x) \to 0$ pointwise for all $x > 0$ as $\epsilon \to 0^+$. (Of course, one could choose ϵ_T in a way that the left hand side is much smaller than $1/\log T$, if desired.)

On the other hand from Proposition 5.10,

$$\begin{aligned}\widetilde{G}_T\big((f|_{\epsilon_T})\hat{},t\big) &= \frac{-1}{\log T}\int_{-\infty}^{\infty}(f|_{\epsilon_T})\Big(\tfrac{x}{\log T}\Big)e^{ixt} + (f|_{\epsilon_T})\Big(\tfrac{-x}{\log T}\Big)e^{-ixt}\,dz(x)\\ &= \frac{-1}{\log T}\int_{-\infty}^{\infty}(f|_{\epsilon_T})\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x) + O_f\Big(\tfrac{1}{\log T}\Big).\end{aligned}$$

Combining these equations gives (56), and (57) can be proved the same way (or alternatively, by conjugation). Note that we may suppose $\epsilon_T \to 0$, and if (56) and (57) hold true for some ϵ_T, they also hold true for any ϵ_T' with $\epsilon_T' \leq \epsilon_T$.

We also have for admissible $f_1, ..., f_j$, $g_1, ..., g_k$, with $f := f_1 \otimes \cdots \otimes f_j$, $g := g_1 \otimes \cdots \otimes g_k$,

$$\begin{aligned}\Psi_T(f;g) &= \frac{1}{\log^{j+k}T}\int_{\mathbb{R}^k}\int_{\mathbb{R}^j} f\Big(\tfrac{x}{\log T}\Big)g\Big(\tfrac{y}{\log T}\Big)\hat{\sigma_1}\big(\tfrac{T}{2\pi}(x_1+\cdots+x_k-y_1-\cdots-y_k)\big)\,d^jz(x)\,d^kz(y)\\ &= \frac{1}{\log^{j+k}T}\int_{\mathbb{R}}\frac{\sigma_1(t/T)}{T}\bigg(\prod_{\ell=1}^{j}\int_{-\infty}^{\infty} f\Big(\tfrac{x_\ell}{\log T}\Big)e^{ix_\ell t}\,dz(x_\ell)\prod_{\ell'=1}^{k}\int_{-\infty}^{\infty} g_{\ell'}\Big(\tfrac{y_\ell}{\log T}\Big)e^{-iy_\ell}\,dz(y_\ell)\bigg)\,dt.\end{aligned}$$

We are above able to interchange the order of integrations in the variable t or other variables as an application of Fubini's theorem because for fixed T and any admissible function f_ℓ (or $g_{\ell'}$) above,

$$\int_{\mathbb{R}}\Big|f_\ell\Big(\tfrac{x}{\log T}\Big)\Big|\,d\big(\psi(e^x)+e^x\big) < +\infty.$$

Hence from this representation of Ψ_T and (56) and (57), there is some $\epsilon_T \to 0+$ such that

$$\begin{aligned}\Psi_T(f;g) &= (-1)^{j+k}\int_{\mathbb{R}}\frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\Big(\widetilde{G}_T\big((f_\ell|_{\epsilon_T})\hat{},t\big)+O_f\big(\tfrac{1}{\log T}\big)\Big)\prod_{\ell'=1}^{k}\Big(\widetilde{G}_T\big((g_{\ell'}|_{\epsilon_T})\check{},t\big)+O_g\big(\tfrac{1}{\log T}\big)\Big)\,dt\\ &= (-1)^{j+k}\int_{\mathbb{R}}\frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_{\epsilon_T})\hat{},t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((g_{\ell'}|_{\epsilon_T})\check{},t\big)\,dt + O_{f,g}\Big(\tfrac{1}{\log T}\Big),\end{aligned} \tag{58}$$

the second line following from expanding the product in the first, and using Hölder's inequality and Lemma 10.1 to bound those terms in which an error term appears.

We will show shortly that for all $\epsilon > \rho > 0$,

$$\begin{aligned}&\int_{\mathbb{R}}\frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_\rho)\hat{},t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((g_{\ell'}|_\rho)\check{},t\big)\,dt \qquad (59)\\ &= \int_{\mathbb{R}}\frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{k}\widetilde{G}_T\big((f_\ell|_\epsilon)\hat{},t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((g_{\ell'}|_\epsilon)\check{},t\big)\,dt + O_{f,g}(\epsilon),\end{aligned}$$

and likewise that

$$\mathop{\mathbf{E}}_{\mathcal{S}'_N}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\rho)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\rho)\check{}(x_i)\Big) \tag{60}$$

$$= \mathop{\mathbf{E}}_{\mathcal{S}'_N}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(x_i)\Big) + O_{f,g}(\epsilon),$$

Let us for the moment assume the truth of these bounds (59) and (60) to see that they allow us to derive identity (12) on the GUE Conjecture. From (58) and (59), with $\rho = \epsilon_T$, for any $\epsilon > 0$, for sufficiently large T,

$$\Psi_T(f;g) = (-1)^{j+k}\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_\epsilon)\hat{},t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((g_{\ell'}|_\epsilon)\check{},t\big)\,dt + O_{f,g}(\epsilon). \tag{61}$$

But from (34), because

$$\int_{\mathbb{R}} (f_\ell|_\epsilon)\hat{}(\alpha)\,d\alpha = \int_{\mathbb{R}} (g_{\ell'}|_\epsilon)\check{}(\alpha)\,d\alpha = 0$$

for all ℓ, ℓ', we have

$$\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_\epsilon)\hat{},t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((g_{\ell'}|_\epsilon)\check{},t\big)\,dt$$

$$= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}\Big(G_T\big((f_\ell|_\epsilon)\hat{},t\big) + O_{f,\epsilon}\Big(\tfrac{1}{\log T}\Big)\Big)\prod_{\ell'=1}^{k}\Big(G_T\big((g_{\ell'}|_\epsilon)\check{},t\big) + O_{g,\epsilon}\Big(\tfrac{1}{\log T}\Big)\Big)\,dt$$

$$= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^{j}G_T\big((f_\ell|_\epsilon)\hat{},t\big)\prod_{\ell'=1}^{k}G_T\big((g_{\ell'}|_\epsilon)\check{},t\big)\,dt + O_{f,g,\epsilon}\Big(\tfrac{1}{\log T}\Big),$$

using in the last step the Fujii upper bound, Lemma 7.2. (The last line could also be obtained from the GUE Conjecture itself, since we are at this point assuming it.) We substitute this in equation (61).

In the language of point processes what we have thus shown, by assuming (59) for the moment, is that

$$\Psi_T(f;g) = \mathop{\mathbf{E}}_{Z_T(\sigma_1)}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(\xi_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(\xi_i)\Big) + O_{f,g}(\epsilon) + O_{f,g,\epsilon}\Big(\tfrac{1}{\log T}\Big). \tag{62}$$

GUE(σ_1) implies that

$$\lim_{T\to\infty}\mathop{\mathbf{E}}_{Z_T(\sigma_1)}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(\xi_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(\xi_i)\Big)$$

$$= \mathop{\mathbf{E}}_{\mathcal{S}}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(x_i)\Big).$$

In particular, because ϵ is arbitrary in (62), $\Psi_T(f;g)$ has a limit as $T\to\infty$ for admissible f, g.

But in turn (from Proposition 5.8),

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\mathcal{S}}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(x_i)\Big)\\
&=\lim_{N\to\infty}\mathop{\mathbf{E}}_{\mathcal{S}'_N}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(x_i)\Big).
\end{aligned}$$

Note that, for any $\epsilon_N > 0$,

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\mathcal{S}'_N}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_{\epsilon_N})\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_{\epsilon_N})\check{}(x_i)\Big)\\
&=\frac{1}{N^{j+k}}\sum_{r\in\mathbb{Z}^j}\sum_{s\in\mathbb{Z}^k}\prod_{\ell=1}^{j} f_\ell|_{\epsilon_N}\Big(\tfrac{-r_\ell}{N}\Big)\prod_{\ell'=1}^{k} g_{\ell'}|_{\epsilon_N}\Big(\tfrac{s_{\ell'}}{N}\Big)\int_{\mathcal{U}(N)}\prod_{\ell=1}^{j}\mathrm{Tr}(u^{r_\ell})\prod_{\ell'=1}^{k}\mathrm{Tr}(u^{s_{\ell'}})\,du\\
&=\frac{1}{N^{j+k}}\sum_{r\in\mathbb{N}_+^j}\sum_{s\in\mathbb{N}_+^k}\prod_{\ell=1}^{j} f_\ell|_{\epsilon_N}\Big(\tfrac{r_\ell}{N}\Big)\prod_{\ell'=1}^{k} g_{\ell'}|_{\epsilon_N}\Big(\tfrac{s_{\ell'}}{N}\Big)\int_{\mathcal{U}(N)}\prod_{\ell=1}^{j}\mathrm{Tr}(u^{-r_\ell})\prod_{\ell'=1}^{k}\mathrm{Tr}(u^{s_{\ell'}})\,du\\
&=\frac{1}{N^{j+k}}\sum_{r\in\mathbb{N}_+^j}\sum_{s\in\mathbb{N}_+^k}\prod_{\ell=1}^{j} f_\ell|_{\epsilon_N}\Big(\tfrac{r_\ell}{N}\Big)\prod_{\ell'=1}^{k} g_{\ell'}|_{\epsilon_N}\Big(\tfrac{s_{\ell'}}{N}\Big)\int_{\mathcal{U}(N)}\prod_{\ell=1}^{j}\mathrm{Tr}(u^{r_\ell})\prod_{\ell'=1}^{k}\mathrm{Tr}(u^{-s_{\ell'}})\,du, \quad (63)
\end{aligned}$$

using Proposition 5.9 (that mixed moments of traces are real valued) in the last line.

For any function f, for $\epsilon_N \le 1/N$,

$$f|_{\epsilon_N}(r/N) = f(r/N)$$

for any positive integer r. Therefore for such ϵ_N, (63) is just

$$(-1)^{j+k}\Theta_N(f;g).$$

Letting ϵ be arbitrary, and using $\rho = \epsilon_N$ in (60), we see, in the same way as for Ψ_T, that $\Theta_N(f;g)$ has a limit as $N\to\infty$. But for any $\epsilon > 0$, both limits will be within $O_{f,g}(\epsilon)$ of

$$(-1)^{j+k}\mathop{\mathbf{E}}_{\mathcal{S}}\Big(\prod_{\ell=1}^{j}\sum_i (f_\ell|_\epsilon)\hat{}(x_i)\Big)\Big(\prod_{\ell'=1}^{k}\sum_i (g_{\ell'}|_\epsilon)\check{}(x_i)\Big)$$

and therefore $O_{f,g}(\epsilon)$ of each other. Because ϵ is arbitrary this is (12).

We therefore need only verify (59) and (60).

To verify (59), note that

$$\begin{aligned}
\widetilde{G}_T\big((f_\ell|_\rho)\hat{}, t\big) &= \widetilde{G}_T\big((f_\ell|_\rho^\epsilon)\hat{}, t\big) + \widetilde{G}_T\big((f_\ell|_\epsilon)\hat{}, t\big)\\
&:= a_\ell + A_\ell.
\end{aligned}$$

In addition to this shorthand, we also use

$$\begin{aligned}
b_{\ell'} &:= \widetilde{G}_T\big((g_{\ell'}|_\rho^\epsilon)\check{}, t\big)\\
B_{\ell'} &:= \widetilde{G}_T\big((g_{\ell'}|_\epsilon)\check{}, t\big).
\end{aligned}$$

Substituted into (59), we show that the terms $a_\ell, b_{\ell'}$ make a small contribution. More exactly,

$$\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{j} \widetilde{G}_T\big((f_\ell|_\rho)\hat{\ }, t\big) \prod_{\ell'=1}^{k} \widetilde{G}_T\big((g_{\ell'}|_\rho)\check{\ }, t\big)\, dt \tag{64}$$
$$= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{j} (a_\ell + A_\ell) \prod_{\ell'=1}^{k} (b_{\ell'} + B_{\ell'})\, dt$$
$$= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{j} A_\ell \prod_{\ell'=1}^{k} B_{\ell'}\, dt$$
$$+ \sum_{\substack{\emptyset \subseteq J \subseteq [j] \\ \emptyset \subseteq K \subseteq [k] \\ J \cup K \neq \emptyset}} \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell \in J} a_\ell \prod_{\lambda \notin J} A_\lambda \prod_{\ell' \in K} b_{\ell'} \prod_{\lambda' \notin K} B_{\lambda'}\, dt.$$

But for any of the terms in this last sum, by Hölder's inequality,

$$\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell \in J} a_\ell \prod_{\lambda \notin J} A_\lambda \prod_{\ell' \in K} b_{\ell'} \prod_{\lambda' \notin K} B_{\lambda'}\, dt$$
$$\leq \prod_{\ell \in J} \Big(\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} |a_\ell|^{j+k}\, dt \Big)^{1/(j+k)} \prod_{\lambda \notin J} \Big(\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} |A_\lambda|^{j+k}\, dt \Big)^{1/(j+k)}$$
$$\times \prod_{\ell' \in K} \Big(\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} |b_{\ell'}|^{j+k}\, dt \Big)^{1/(j+k)} \prod_{\lambda' \notin K} \Big(\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} |B_{\lambda'}|^{j+k}\, dt \Big)^{1/(j+k)}$$
$$= \prod_{\ell \in J} O_{f_\ell}(\epsilon) \prod_{\lambda \notin J} O_{f_\ell}(1) \prod_{\ell' \in K} O_{g_{\ell'}}(\epsilon) \prod_{\lambda' \notin K} O_{g_{\lambda'}}(1),$$

for sufficiently large T. Here we have used Lemma 7.2 (the Fujii bound) to bound those terms with A_λ or $B_{\lambda'}$, and Corollary 7.7 to bound those terms with a_ℓ or b'_ℓ, recalling that $f_\ell|_\rho^\epsilon$ and $g_{\ell'}|_\rho^\epsilon$ are supported in the interval $[0, \epsilon]$.

In no term of the finite sum in the last line of (64) are both J and K empty, and so (64) is just

$$\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{j} A_\ell \prod_{\ell'=1}^{k} B_{\ell'}\, dt + O_{f,g}(\epsilon).$$

This demonstrates (59).

(60) is proven in the same way, substituting Proposition 5.8 for Fujii's upper bound, and Corollary 8.4 for Corollary 7.7. □

11.3. A proof in the opposite direction is less technically demanding.

Proof of Theorem 2.4: (12) implies the GUE Conjecture. Assume that (12) holds for all admissible functions. Let $f_1, ..., f_n$ be arbitrary $C_c^2(\mathbb{R})$ functions. From (12), we have for any $\{\varepsilon_1, ..., \varepsilon_n\} \in \{-1, 1\}^n$,

$$\lim_{T\to\infty} \frac{(-1)^n}{\log^n T} \int_{\mathbb{R}^n} f_1\Big(\tfrac{\varepsilon_1 x_1}{\log T}\Big) \cdots f_n\Big(\tfrac{\varepsilon_n x_n}{\log T}\Big) \big(1 - T|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n|\big)_+ \, d^n z(x)$$
$$= \lim_{N\to\infty} \frac{1}{N^n} \sum_{r \in \mathbb{N}_+^n} f_1\Big(\tfrac{\varepsilon_1 r_1}{N}\Big) \cdots f_n\Big(\tfrac{\varepsilon_n r_n}{N}\Big) \int_{\mathcal{U}(N)} \prod_{\ell=1}^{n} \mathrm{Tr}(u^{\epsilon_n r_n})\, du.$$

But by the explicit formula,

$$\begin{aligned}
&\sum_{\varepsilon\in\{-1,1\}^n} \frac{(-1)^n}{\log^n T}\int_{\mathbb{R}^n} f_1\Big(\tfrac{\varepsilon_1 x_1}{\log T}\Big)\cdots f_n\Big(\tfrac{\varepsilon_n x_n}{\log T}\Big)\big(1-T|\varepsilon_1 x_1+\cdots+\varepsilon_n x_n|\big)_+\, d^n z(x)\\
&= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^n \tilde{G}_T(\hat{f}_\ell, t)\, dt.
\end{aligned}$$

From Stirling's formula, in particular (34), this is equal as $T\to\infty$ to

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^n \Big(G_T(\hat{f}_\ell,t) - \frac{\log(|t|+2)}{\log T}\int_{\mathbb{R}}\hat{f}_\ell(\alpha)\,d\alpha + O_{f_\ell}\Big(\tfrac{1}{\log T}\Big)\Big)\,dt\\
&= \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T}\prod_{\ell=1}^n \Big(G_T(\hat{f}_\ell,t) - \frac{\log(|t|+2)}{\log T}\int_{\mathbb{R}}\hat{f}_\ell(\alpha)\,d\alpha\Big)\,dt + O_f\Big(\tfrac{1}{\log T}\Big), \qquad (65)
\end{aligned}$$

the last line following from Lemma 7.2 (the Fujii bound) in the same manner we have used it previously. Because we have

$$\frac{\log(|t|+2)}{\log T} = 1 + O\Big(\frac{|\log\big(\frac{|t|}{T}+\frac{2}{T}\big)|}{\log T}\Big)$$

we may use Lemma 7.2 once again so see that the expression in (65) is equal to

$$\mathop{\mathbf{E}}_{Z_t(\sigma_1)}\prod_{\ell=1}^n\Big(\sum_i \hat{f}_\ell(\xi_i) - \int_{\mathbb{R}}\hat{f}_\ell(\alpha)\,d\alpha\Big) + O_f\Big(\tfrac{1}{\log T}\Big).$$

On the other hand,

$$\begin{aligned}
&\sum_{\varepsilon\in\{-1,1\}^n}\frac{1}{N^n}\sum_{r\in\mathbb{N}_+^n} f_1\Big(\tfrac{\varepsilon_1 r_1}{N}\Big)\cdots f_n\Big(\tfrac{\varepsilon_n r_n}{N}\Big)\int_{\mathcal{U}(N)}\prod_{\ell=1}^n \mathrm{Tr}(u^{\epsilon_n r_n})\,du\\
&= \mathop{\mathbf{E}}_{\mathcal{S}'_N}\prod_{\ell=1}^n\Big(\sum_i \hat{f}_\ell(x_i) - \int_{\mathbb{R}}\hat{f}_\ell(\alpha)\,d\alpha\Big)
\end{aligned}$$

by equation (30).

Thus, it inductively follows (by removing lower order correlations) that for $f_1, ..., f_n$ arbitrary $C_c^2(\mathbb{R})$ functions

$$\begin{aligned}
\lim_{T\to\infty}\mathop{\mathbf{E}}_{Z_T(\sigma)}\sum_{\substack{j_1,\dots,j_n\\ \text{distinct}}}\hat{f}_1(\xi_{j_1})\cdots\hat{f}_n(\xi_{j_n}) &= \lim_{N\to\infty}\mathop{\mathbf{E}}_{\mathcal{S}'_N}\sum_{\substack{j_1,\dots,j_n\\ \text{distinct}}}\hat{f}_1(x_1)\cdots\hat{f}_n(x_n)\\
&= \mathop{\mathbf{E}}_{\mathcal{S}}\sum_{\substack{j_1,\dots,j_n\\ \text{distinct}}}\hat{f}_1(x_1)\cdots\hat{f}_n(x_n). \qquad (66)
\end{aligned}$$

Yet, any such $\hat{f}_1\otimes\cdots\otimes\hat{f}_n$ will decay quadratically in each variable, and if $\mathcal{A}_n$ is the linear span of such functions:

$$\mathcal{A}_n := \mathrm{span}\{\eta\colon \mathbb{R}^n\to\mathbb{R} : \eta = \hat{f}_1\otimes\cdots\otimes\hat{f}_n,\ f_1,...,f_n\in C_c^2(\mathbb{R})\},$$

it is easy to see that for any $\rho\in C_c(\mathbb{R}^k)$ and any $\epsilon>0$, there exists $\eta\in\mathcal{A}_n$ so that for all x,

$$\big|\rho(x)-\eta(x)\big| \le \frac{\epsilon}{(1+|x_1|^{3/2})\cdots(1+|x_n|^{3/2})}. \qquad (67)$$

For, using (51), for any $\eta \in C_c(\mathbb{R})$ and $\epsilon > 0$, there exists $f \in C_c^2(\mathbb{R})$ such that for all x

$$|\eta(x) - \hat{f}(x)| \leq \frac{\epsilon}{1 + |x|^{3/2}}.$$

And quite generally if $\mathcal{B}$ is dense in $C_c(\mathbb{R})$, then the linear span of functions $\{\eta_1 \otimes \cdots \otimes \eta_k : \eta_j \in \mathcal{B} \ \ \forall j\}$ is dense in $C_c(\mathbb{R}^k)$. In the case that $\mathcal{B} = \{(1 + |x|^{3/2})\hat{f}(x) : f \in C_c^2(\mathbb{R})\}$ this yields (67).

Therefore, because $\mathcal{A}_n$ is in this sense sufficiently dense, by Proposition 9.4, (66) is sufficient to deduce GUE(σ), and therefore the GUE Conjecture proper. □

11.4. Note that in the above proofs to pass from (12) to the GUE Conjecture and back, we did not require knowledge of correlation functions at all levels, but rather for any n, knowing the first n correlation functions of the zeta zeros was sufficient to pass to (12) for all $j + k \leq n$, and vice versa.

Because we know the $n = 1$ case of the GUE Conjecture is true unconditionally, we have as a corollary to Theorem 2.4 an arithmetic statement that is equivalent to the pair correlation conjecture.

Corollary 11.1. *The case $n = 2$ of the GUE Conjecture is equivalent to the claim that for all admissible $f, g\colon \mathbb{R} \to \mathbb{R}$,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f\Big(\tfrac{x}{\log T}\Big) g\Big(\tfrac{y}{\log T}\Big) \upsilon_T(x,y)\, dz(x) dz(y) = \log^2 T \Big(\int_0^\infty f(\alpha) g(\alpha) (\alpha \wedge 1)\, d\alpha + o(1) \Big).$$

On the right hand side,

$$\int_0^\infty f(\alpha) g(\alpha)(\alpha \wedge 1)\, d\alpha = \frac{1}{N^2} \sum_{r,s=1}^{\infty} f\Big(\tfrac{r}{N}\Big) g\Big(\tfrac{s}{N}\Big) \int_{\mathcal{U}(N)} \mathrm{Tr}(u^r) \overline{\mathrm{Tr}(u^s)}\, du,$$

which can be seen from either the Diaconis-Shashahani type identity (17) or the explicit calculation of correlation functions for eigenvalues of $\mathcal{U}(N)$, Theorem 5.6. The latter approach is somewhat more tedious, involving as it does an inclusion-exclusion argument, but for us it will generalize.

We have outlined in section 1 how Corollary 11.1 reduces to Theorem 1.2, a weighted estimate for the variance of primes in short intervals, with an algebraically nice form. We record below the analogues of Corollary 11.1 for the cases $n = 3, 4$, but the resulting statements are less simple than Theorem 1.2.

On the other hand, we do derive a generalization of Theorem 1.2 which is algebraically simple in section 13. This is the covariance of almost primes with higher order von Mangoldt weights. The estimates we consider there fall short however of implying in full that any n-level densities for the zeta zeros follow the GUE pattern, beyond $n = 2$.

Corollary 11.2 (The three point correlation conjecture). *Assume the pair correlation conjecture, that* (1) *holds for $n = 2$ for all fixed η.*

Then the statement that (1) *holds for $n = 3$ for all η is equivalent to the statement that for all admissible f_1, f_2, g*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} f_1\Big(\tfrac{x_1}{\log T}\Big) f_2\Big(\tfrac{x_2}{\log T}\Big) g\Big(\tfrac{y}{\log T}\Big) \upsilon_T(x_1 + x_2, y)\, dz(x_1) dz(x_2)\, dz(y) \tag{68}$$

$$= \log^3 T\Big(\int_{\mathbb{R}_+^2} f_1(\alpha_1) f_2(\alpha_2) g(\alpha_1 + \alpha_2) \big[(\alpha_1 \wedge 1) + (\alpha_2 \wedge 1) - ((\alpha_1 + \alpha_2) \wedge 1)\big]\, d\alpha_1 d\alpha_2 + o(1)\Big).$$

Corollary 11.3 (The four point correlation conjecture). *Assume the pair correlation conjecture and the three point correlation conjectures, that is, that* (1) *holds for $n = 2$ and 3 for all fixed η.*

Then the statement that (1) *holds for $n = 4$ for all η is equivalent to the claim that both:*

(i) For all admissible f_1, f_2, f_3, g,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} f_1\Big(\tfrac{x_1}{\log T}\Big) f_2\Big(\tfrac{x_2}{\log T}\Big) f_3\Big(\tfrac{x_3}{\log T}\Big) g\Big(\tfrac{y}{\log T}\Big) \upsilon_T(x_1 + x_2 + x_3, y)\, dz(x_1) dz(x_2) d(x_3)\, dz(y)$$

$$= \log^4 \Big(T \int_{\mathbb{R}_+^3} f_1(\alpha_1) f_2(\alpha_2) f_3(\alpha_3) g(\alpha_1 + \alpha_2 + \alpha_3) \big[(\alpha_1 \wedge 1) + (\alpha_2 \wedge 1) + (\alpha_3 \wedge 1) \tag{69}$$

$$- ((\alpha_1 + \alpha_2) \wedge 1) - ((\alpha_1 + \alpha_3) \wedge 1) - ((\alpha_2 + \alpha_3) \wedge 1) + ((\alpha_1 + \alpha_2 + \alpha_3) \wedge 1)\big]\, d\alpha_1 d\alpha_2 d\alpha_3 + o(1)\Big)$$

and

(ii) For all admissible f_1, f_2, g_1, g_2,

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1\Big(\tfrac{x_1}{\log T}\Big) f_2\Big(\tfrac{x_2}{\log T}\Big) g_1\Big(\tfrac{y_1}{\log T}\Big) g\Big(\tfrac{y_2}{\log T}\Big) \upsilon_T(x_1 + x_2, y_1 + y_2)\, dz(x_1) dz(x_2)\, dz(y_1) dz(y_2)$$

$$= \log^4 T\Big(\int_{\mathbb{R}_+^4} f_1(\alpha_1) f_2(\alpha_2) g_1(\beta_1) g_2(\beta_2) \Big[\delta(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)\Big(1 + (1 - \alpha_1)_+ + (1 - \alpha_2)_+$$

$$+ (1 - \beta_1)_+ + (1 - \beta_2)_+ - (1 - \alpha_1 - \alpha_2)_+ - (1 - |\alpha_1 - \beta_1|)_+ - (1 - |\alpha_1 - \beta_2|)_+$$

$$- 2(1 - \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2)_+\Big) + \delta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)(\alpha_1 \wedge 1)(\alpha_2 \wedge 1) \tag{70}$$

$$+ \delta(\alpha_1 - \beta_2)\delta(\alpha_2 - \beta_1)(\alpha_1 \wedge 1)(\alpha_2 \wedge 1)\Big]\, d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 + o(1)\Big).$$

One can of course continue on in this way for even higher correlations.

12. Zeros and the zeta function

12.1. We turn to upper bounds for moments of ζ'/ζ.

Proof of Theorem 2.1. We recall Theorem 4.2, that for $f(x) = \exp(-Ax)$, and $\log T \geq 2A$,

$$\frac{1}{\log T} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} - it\Big) = \frac{-1}{\log T} \int_{\to -\infty}^{\to\infty} f\Big(\tfrac{x}{\log T}\Big) e^{ixt}\, dz(x)$$

$$= O_f\Big(\tfrac{1}{\log T}\Big) + \frac{-1}{\log T} \int_0^{\to\infty} f\Big(\tfrac{x}{\log T}\Big) e^{ixt}\, dz(x).$$

There exists ϵ_T close enough to 0 so that this expression is

$$O_f\Big(\tfrac{1}{\log T}\Big) + \frac{-1}{\log T}\int_0^{\to\infty} f|_{\epsilon_T}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x)$$
$$= O_f\Big(\tfrac{1}{\log T}\Big) + \lim_{R\to\infty}\frac{-1}{\log T}\int_{\mathbb{R}} f|_{\epsilon_T}^{R}\Big(\tfrac{x}{\log T}\Big)e^{ixt}\,dz(x),$$

the second line being an easy exercise. Using Proposition 5.10, this is

$$O_f\Big(\tfrac{1}{\log T}\Big) + \lim_{R\infty}\widetilde{G}_T\big((f|_{\epsilon_T}^{R})\hat{}, t\big).$$

Because

$$\lim_{R\to\infty}\sup_{R'>R}\|f|_R^{R'}\|_{L^1(\mathbb{R})} = 0$$
$$\lim_{R\to\infty}\sup_{R'>R}\|(f|_R^{R'})''\|_{L^1(\mathbb{R})} = 0$$

we have from (35) and (51) that for any $\delta > 0$, there exists R_δ so that

$$\lim_{R\to\infty}\big|\widetilde{G}_T\big((|_{R_\delta}^{R})\hat{}, t\big)\big| \le \delta\log(|t|+2).$$

In particular, setting $\delta = 1/T$, we see that

$$\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} - it\Big) = \widetilde{G}_T\big((f|_{\epsilon_T}^{R_{1/T}})\hat{}, t\big) + O_f\Big(\tfrac{1}{\log T}\Big) + O\Big(\tfrac{\log(|t|+2)}{T}\Big)$$
$$= \widetilde{G}_T\big((f_{\epsilon_T}^{1/k})\hat{}, t\big) + \widetilde{G}_T\big((f|_{1/k}^{R_{1/T}})\hat{}, t\big) + O_f\Big(\tfrac{1}{\log T}\Big) + O\Big(\tfrac{\log(|t|+2)}{T}\Big),$$

so from Minkowski's inequality,

$$\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\Big|\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} - it\Big)\Big|^k dt \lesssim_k \int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\big|\widetilde{G}_T\big((f_{\epsilon_T}^{1/k})\hat{}, t\big)\big|^k dt \tag{71}$$
$$+ \int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\big|\widetilde{G}_T\big((f|_{1/k}^{R_{1/T}})\hat{}, t\big)\big|^k dt + o_f(1),$$

where in this case we define $\sigma(t)$ to be $\mathbf{1}_{[1,2]}(t)$ (though one could certainly use other weights). From Lemma 7.4 (or Lemma 7.7), because $f_{\epsilon_T}^{1/k}$ is supported in $[0, 1/k]$,

$$\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\big|\widetilde{G}_T\big((f_{\epsilon_T}^{1/k})\hat{}, t\big)\big|^k dt = O_{f,k}(1).$$

Likewise, because

$$\|f_{1/k}^{R_{1/t}}\|_{L^1(\mathbb{R})} = O_{f,k}(1)$$

$$\|(f_{1/k}^{R_{1/t}})''\|_{L^1(\mathbb{R})} = O_{f,k}(1)$$

as $T \to \infty$, Lemma 10.1 implies that

$$\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\big|\widetilde{G}_T\big((f|_{1/k}^{R_{1/T}})\hat{}, t\big)\big|^k dt = O_{f,k}(1).$$

Substituting these bounds in (71) gives the theorem. □

Remark: This analysis can be reproduced in a more traditional way by using the famous Selberg mollification formula:

$$\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_x(n)}{n^s} + \frac{1}{\log x} \sum_{\gamma} \frac{x^{1/2+i\gamma-s} - x^{2(1/2+i\gamma-s)}}{(1/2+i\gamma-s)^2} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x} \sum_{q=1}^{\infty} \frac{x^{-2q-s} - x^{-2(2q+s)}}{(2q+s)^2},$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \le n \le x \\ \Lambda(n) \frac{\log(x^2/n)}{\log x} & \text{for } x \le n \le x^2 \\ 0 & \text{otherwise.} \end{cases}.$$

Letting $x = \frac{1}{2k} \log T$ and $s = \frac{1}{2} + \frac{A}{\log T} + it$, we can produce an upper bound for

$$\frac{1}{T} \int_T^{2T} \left| \frac{1}{\log T} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big) \right|^k dt$$

in the same way as above, with

$$\frac{1}{\log T} \left(\sum \frac{\Lambda_x(n)}{n^s} \right) \quad \text{playing the role of} \quad \widetilde{G}_T\big((f_{\epsilon_T}^{1/k})\hat{\ }, t\big),$$

and

$$\frac{1}{\log T} \frac{1}{\log x} \sum_{\gamma} \frac{x^{1/2+i\gamma-s} - x^{2(1/2+i\gamma-s)}}{(1/2+i\gamma-s)^2} \quad \text{playing the role of} \quad \widetilde{G}_T\big((f|_{1/k}^{R_{1/T}})\hat{\ }, t\big),$$

and the remaining terms of order $O(1)$. Indeed, the latter sum over γ can be bounded from above by $G_T(\eta, t)$ for some η of quadratic decay, moments of which can be bounded with Fujii's theorem.

Remark: On the surface this upper bound may seem to resemble upper bounds for the moments of the zeta function itself on the critical line. (See [**60**], [**50**], [**29**] for recent conditional results of this sort.) The resemblance is somewhat superficial however; as we have shown, the moments of $\zeta'(s)/\zeta(s)$ at a microscopic distance from the critical line concern only microscopic interactions between zeros, while the moments of $\zeta(s)$ have a much more global dependence, influenced – perhaps especially in their arithmetic factor – by macroscopic statistical interactions among zeros.

12.2. We can now turn to a proof of Theorem 2.2, our restatement of the GUE Conjecture in terms of the mixed moments of the zeta function. In this subsection we demonstrate the asymptotic equality of (8) and (9) on the assumption of the GUE Conjecture. In fact, we show more:

Theorem 12.1. *For $1 \le j \le j$ and $1 \le \ell' \le k$, define*

$$f_\ell(x) := P_\ell(x) e^{-A_\ell x}$$

$$g_{\ell'}(x) := Q_{\ell'}(x) e^{-B_{\ell'} x}$$

where P_ℓ and $Q_{\ell'}$ are polynomials and $A_\ell, B_{\ell'}$ are constants with $\Re A_\ell, \Re B_{\ell'} > 0$. Let $\sigma(t)$ be either the function $\mathbf{1}_{[1,2]}$ or $\frac{1}{2\pi}(\sin(t/2)/(t/2))^2$ as in (50). Then the GUE Conjecture *implies that*

$$\lim_{T\to\infty}\frac{1}{\log^{j+k}T}\int_{\mathbb{R}}\frac{\sigma(t/T)}{T}\left(\prod_{\ell=1}^{k}\int_{\to\infty}^{\to\infty} f_\ell\big(\tfrac{x_\ell}{\log T}\big)e^{-ix_\ell t}\,dz(x_\ell)\prod_{\ell=1}^{k}\overline{\int_{\to-\infty}^{\to\infty} g_{\ell'}\big(\tfrac{x_{\ell'}}{\log T}\big)e^{-ix_{\ell'} t}\,dz(x_{\ell'})}\right)dt$$
$$=\lim_{N\to\infty}\Theta_N(f_1\otimes\cdots\otimes f_j;\overline{g}_1\otimes\cdots\otimes\overline{g}_k). \tag{72}$$

Remark: It follows from partial integration as before that the integrals

$$\int_{\to-\infty}^{\to\infty} f_\ell\Big(\tfrac{x}{\log T}\Big)\,e^{-ixt}\,dz(x)$$

converge.

In the case that $f_\ell = g_{\ell'} = 1$, we see from Theorem 4.2 that the left hand side of equation (72) is exactly $(-1)^{j+k}$ times the expression (8) in Theorem 2.2, while the right hand side is

$$\lim_{N\to\infty}\frac{1}{N^{j+k}}\sum_{r\in\mathbb{N}_+^j}\sum_{s\in\mathbb{N}_+^k}\left(\prod_{\ell=1}^{j}e^{-A_\ell r_\ell/N}\prod_{\ell'=1}^{k}\overline{e^{-B_{\ell'}s_{\ell'}/N}}\right)$$
$$\times\left(\int_{\mathcal{U}(N)}\prod_{\ell=1}^{j}(-\mathrm{Tr}\,u^{r_\ell})\prod_{\ell'=1}^{k}\overline{(-\mathrm{Tr}\,u^{s_{\ell'}})}\,du\right)$$
$$=\lim_{N\to\infty}(-1)^{j+k}\int_{\mathcal{U}(N)}\prod_{\ell=1}^{j}\frac{Z'}{Z}\Big(\frac{A_\ell}{N}\Big)\prod_{\ell'=1}^{k}\overline{\frac{Z'}{Z}\Big(\frac{B_{\ell'}}{N}\Big)}\,du,$$

where we can swap the order of integration and summation because for fixed N, $\mathrm{Tr}(u^r)$ is bounded as $r\to\infty$. This is, of course $(-1)^{j+k}$ times expression (9).

More generally, if $f(x) = P(x)e^{-Ax}$ for a polynomial P, note that

$$\sum_{r=1}^{\infty} f\Big(\tfrac{r}{N}\Big)\mathrm{Tr}(u^r) = P\Big(\frac{d}{dA}\Big)\frac{Z'}{Z}\Big(\frac{A}{N}\Big), \tag{73}$$

and likewise for the zeta function,

$$\int_{\to-\infty}^{\to\infty} f\Big(\tfrac{x}{\log T}\Big)e^{-ixt}\,dz(x) = P\Big(\frac{d}{dA}\Big)\frac{\zeta'}{\zeta}\Big(\tfrac12+\tfrac{A}{\log T}+it\Big). \tag{74}$$

We will use this more general framework when proving Theorem 13.3 and evaluating the covariance of almost primes.

Proof of Theorem 12.1. Let $\epsilon>0$. We have as in the previous subsection that there exists ϵ_T so that

$$\frac{-1}{\log T}\int_{\to-\infty}^{\to\infty} f_\ell\Big(\tfrac{x}{\log T}\Big)e^{-ixt}\,dz(x) = O_f\Big(\tfrac{1}{\log T}\Big)+\widetilde{G}_T\big((f_\ell|_{\epsilon_T}^{\epsilon})\check{},t\big)+\lim_{R\to\infty}\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R})\check{},t\big),$$

and likewise for $g_{\ell'}$. As before, there is some $R_{1/T}$ so that

$$\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R_{1/T}})\check{},t\big) = \lim_{R\to\infty}\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R})\check{},t\big)+O\Big(\tfrac{\log(|t|+2)}{T}\Big),$$

and one can find R', depending on ϵ, but not on T, so that as $T\to\infty$,

$$\|f_\ell|_{R'}^{R_{1/T}}\|_{L^1}<\epsilon$$
$$\|(f_\ell|_{R'}^{R_{1/T}})''\|_{L^1}<\epsilon,$$

and therefore, using Lemma 10.2,

$$\int_{\mathbb{R}} \frac{\sigma(t/T)}{T} \left|\widetilde{G}_T\big((f_\ell|_{R'}^{R_{1/T}})\check{}, t\big)\right|^{j+k} \lesssim \epsilon^{j+k}$$

for sufficiently large T.

Therefore we may decompose our integral against dz:

$$\begin{aligned}\frac{-1}{\log T}\int_{\to-\infty}^{\to\infty} f_\ell\Big(\tfrac{x}{\log T}\Big)e^{-ixt}\,dz(x) =& O_f\Big(\tfrac{1}{\log T}\Big) + \widetilde{G}_T\big((f_\ell|_{\epsilon_T}^{\epsilon})\check{}, t\big) + \widetilde{G}_T\big((f_\ell|_{\epsilon}^{R'})\check{}, t\big) \\ &+ \widetilde{G}_T\big((f_\ell|_{R'}^{R_{1/T}})\check{}, t\big) + O\Big(\tfrac{\log(|t|+2)}{T}\Big),\end{aligned}$$

and likewise

$$\begin{aligned}\overline{\frac{-1}{\log T}\int_{\to-\infty}^{\to\infty} g_{\ell'}\Big(\tfrac{x}{\log T}\Big)e^{-ixt}\,dz(x)} =& O_g\Big(\tfrac{1}{\log T}\Big) + \widetilde{G}_T\big((\overline{g}_{\ell'}|_{\epsilon_T}^{\epsilon})\hat{}, t\big) + \widetilde{G}_T\big((\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}, t\big) \\ &+ \widetilde{G}_T\big((\overline{g}_{\ell'}|_{R'}^{R_{1/T}})\hat{}, t\big) + O\Big(\tfrac{\log(|t|+2)}{T}\Big).\end{aligned}$$

Here the terms

$$\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R'})\check{}, t\big) \text{ and } \widetilde{G}_T\big((\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}, t\big)$$

will be the main contributions. Note that in the second equation we have taken a Fourier transform $(\overline{g}\cdots)\hat{}$, as opposed to the inverse Fourier transform $(f\cdots)\check{}$ in the first equation; the reader should check that this is indeed what arises from conjugating the left hand side.

Applying Hölder's inequality to these decompositions as in section 11, as $T\to\infty$,

$$\begin{aligned}&\frac{(-1)^{j+k}}{\log^{j+k} T}\int_{\mathbb{R}} \frac{\sigma(t/T)}{T}\left(\prod_{\ell=1}^{k}\int_{\to\infty}^{\to\infty} f_\ell\big(\tfrac{x_\ell}{\log T}\big)e^{-ix_\ell t}\,dz(x_\ell)\prod_{\ell=1}^{k}\overline{\int_{\to-\infty}^{\to\infty} g_{\ell'}\big(\tfrac{x_{\ell'}}{\log T}\big)e^{-ix_{\ell'} t}\,dz(x_{\ell'})}\right)dt \\ &= \int_{\mathbb{R}} \frac{\sigma(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R'})\check{}, t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}, t\big)\,dt + O_{f,g}(\epsilon) + o_{f,g}(1). \qquad (75)\end{aligned}$$

But from the GUE Conjecture (which implies GUE(σ) for either choice of σ),

$$\begin{aligned}&\int_{\mathbb{R}} \frac{\sigma(t/T)}{T}\prod_{\ell=1}^{j}\widetilde{G}_T\big((f_\ell|_{\epsilon}^{R'})\check{}, t\big)\prod_{\ell'=1}^{k}\widetilde{G}_T\big((\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}, t\big)\,dt \\ &= \mathop{\mathbf{E}}_{\mathcal{S}}\prod_{\ell=1}^{j}\sum_i (f_\ell|_{\epsilon}^{R'})\check{}(\xi_i)\prod_{\ell'=1}^{k}\sum_i(\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}(\xi_i) + o(1). \qquad (76)\end{aligned}$$

Here we have used that $f_\ell|_{\epsilon}^{R'}$ and $\overline{g}_{\ell'}|_{\epsilon}^{R'}$ are each smooth, implying that $(f_\ell|_{\epsilon}^{R'})\check{}$ and $(\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}$ are guaranteed to have (much faster than) quadratic decay, so that Proposition 9.4 applies.

In turn, the right hand side of (76) is the limit as $N\to\infty$ of

$$\begin{aligned}&\mathop{\mathbf{E}}_{\mathcal{S}'_N}\prod_{\ell=1}^{j}\sum_i (f_\ell|_{\epsilon}^{R'})\check{}(x_i)\prod_{\ell'=1}^{k}\sum_i(\overline{g}_{\ell'}|_{\epsilon}^{R'})\hat{}(x_i) \\ &= \mathop{\mathbf{E}}_{\mathcal{S}'_N}\prod_{\ell=1}^{j}\sum_i (f_\ell|_{\epsilon_N})\check{}(x_i)\prod_{\ell'=1}^{k}\sum_i(\overline{g}_{\ell'}|_{\epsilon_N})\hat{}(x_i) + O_{f,g}(\epsilon) + o_{f,g}(1),\end{aligned}$$

for $\epsilon_N = 1/2N$, using the same estimates as above with the point processes $\mathcal{S}'_N$ in place of $Z_T(\sigma_2)$. But by applying Poisson summation (as in equation (30)),

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\mathcal{S}'_N} \prod_{\ell=1}^{j} \sum_i (f_\ell|_{\epsilon_N})\check{}(x_i) \prod_{\ell'=1}^{k} \sum_i (\overline{g}_{\ell'}|_{\epsilon_N})\hat{}(x_i) \\
&= \int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} \Big(\sum_{r_\ell \in \mathbb{Z}} \frac{1}{N} f_\ell|_{\epsilon_N}\Big(\tfrac{r_\ell}{N}\Big) \mathrm{Tr}(u^{r_\ell}) \Big) \prod_{\ell'=1}^{k} \Big(\sum_{s'_\ell \in \mathbb{Z}} \frac{1}{N} \overline{g}_{\ell'}|_{\epsilon_N}\Big(\tfrac{-s_{\ell'}}{N}\Big) \mathrm{Tr}(u^{s_{\ell'}}) \Big)\, du \\
&= \frac{1}{N^{j+k}} \int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} \Big(\sum_{r=1}^{\infty} f_\ell|_{\epsilon_N}\Big(\tfrac{r}{N}\Big) \mathrm{Tr}(u^{r}) \Big) \prod_{\ell'=1}^{k} \Big(\sum_{s'=1}^{\infty} \overline{g}_{\ell'}|_{\epsilon_N}\Big(\tfrac{s'}{N}\Big) \mathrm{Tr}(u^{-s'}) \Big)\, du.
\end{aligned}$$

Interchanging integration and summation is plainly justified, and we see that this is

$$(-1)^{j+k} \Theta_N(f \otimes \cdots \otimes f_j\, ;\, \overline{g}_1 \otimes \cdots \otimes \overline{g}_k).$$

Because for any $\epsilon > 0$ as $N \to \infty$ this is within $O(\epsilon) + o(1)$ of the right hand side of (76), we see that the right hand limit of (72) exists. But in the same way, for any $\epsilon > 0$ as $T \to \infty$ the left hand side of (75) is within $O(\epsilon) + o(1)$ of the right hand side of (76), so that the left hand limit of (72) exists. Therefore the two limits in (72) are within $O(\epsilon)$ of each other for any ϵ and are thus equal. □

12.3. In the converse direction,

Proof of Theorem 2.2: The equivalence of (8) and (9) implies the GUE Conjecture. Naturally, our proof will bear a similarity to the proof above of the second part of Theorem 2.4. We use the formula that for $L > 0$, τ real, and T sufficiently large,

$$\begin{aligned}
\frac{1}{\log T} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{2\pi(L-i\tau)}{\log T} - it\Big) &= -\frac{1}{\log T} \int_{\to -\infty}^{\to \infty} \exp\Big(-\tfrac{2\pi(L-i\tau)}{\log T} x\Big) e^{ixt}\, dz(x) \\
&= -\frac{1}{\log T} \int_{\to -\infty}^{\to \infty} \exp\Big(-\tfrac{2\pi L}{\log T}|x| + \tfrac{i2\pi\tau}{\log T} x\Big) e^{ixt}\, dz(x) \\
&\quad - \frac{1}{\log T} \int_{-\infty}^{0} \exp\Big(-\tfrac{2\pi(L-i\tau)}{\log T} x\Big) e^{ixt} e^{x/2}\, dx \\
&\quad + \frac{1}{\log T} \int_{-\infty}^{0} \exp\Big(\tfrac{2\pi(L+i\tau)}{\log T} x\Big) e^{ixt} e^{x/2}\, dx \\
&= -\frac{1}{\log T} \int_{\to -\infty}^{\to \infty} \exp\Big(-\tfrac{2\pi L}{\log T}|x| + \tfrac{i2\pi\tau}{\log T} x\Big) e^{ixt}\, dz(x) + O\Big(\frac{1}{\log T}\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\log T} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{2\pi(L-i\tau)}{\log T} - it\Big) + \frac{1}{\log T} \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{2\pi(L+i\tau)}{\log T} + it\Big) \\
&= -\frac{1}{\log T} \int_{\to -\infty}^{\to \infty} \Big[\exp\Big(-\frac{2\pi L|x| - i2\pi\tau x}{\log T}\Big) e^{ixt} + \exp\Big(-\frac{2\pi L|x| + i2\pi\tau x}{\log T}\Big) e^{-ixt} \Big]\, dz(x) + O\Big(\frac{1}{\log T}\Big).
\end{aligned}$$

If

$$f(x) := \exp(-2\pi L|x| + i2\pi\tau x)$$

then

$$\hat{f}(\xi) = h_{L,\tau}(\xi) := \frac{1}{L} h\Big(\frac{\xi - \tau}{L}\Big)$$

where

$$h(\xi) := \frac{1}{\pi(1+\xi^2)}.$$

Hence we have by the explicit formula

$$\begin{aligned}&-\frac{1}{\log T}\int_{\to -\infty}^{\to\infty}\Big[\exp\Big(-\frac{2\pi L|x| - i2\pi\tau x}{\log T}\Big)e^{ixt} + \exp\Big(-\frac{2\pi L|x| + i2\pi\tau x}{\log T}\Big)e^{-ixt}\Big]\,dz(x)\\ &= \lim_{R\to\infty}\int_{-\infty}^{\infty}\hat{\alpha}_R * h_{L,\tau}\big(\tfrac{\log T}{2\pi}(\xi-\tau)\big)\,dS(\xi)\\ &= \int_{-\infty}^{\infty} h_{L,\tau}\big(\tfrac{\log T}{2\pi}(\xi-t)\big)\,dS(\xi),\end{aligned}$$

the last line following from dominated convergence. Therefore, for positive constants $L_1, ..., L_k$ and real $\tau_1, ..., \tau_k$,

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\prod_{\ell=1}^k \tilde{G}_T(h_{L_\ell,\tau_\ell}, t)\,dt\\ =&\frac{1}{T}\int_T^{2T}\prod_{\ell=1}^k\Big(\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac12 + \tfrac{2\pi(L_\ell - i\tau_\ell)}{\log T} - it\Big) + \frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac12 + \tfrac{2\pi(L_\ell + i\tau_\ell)}{\log T} + it\Big) + O\Big(\frac{1}{\log T}\Big)\Big)\,dt\\ =&\frac{1}{T}\int_T^{2T}\prod_{\ell=1}^k\Big(\frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac12 + \tfrac{2\pi(L_\ell - i\tau_\ell)}{\log T} - it\Big) + \frac{1}{\log T}\frac{\zeta'}{\zeta}\Big(\tfrac12 + \tfrac{2\pi(L_\ell + i\tau_\ell)}{\log T} + it\Big)\Big)\,dt + O\Big(\frac{1}{\log T}\Big),\end{aligned}$$

the last line following from Theorem 2.1.

On the assumption of condition (8) and (9), this is asymptotic to

$$Q := \lim_{N\to\infty}\frac{1}{N^k}\int_{\mathcal{U}(N)}\prod_{\ell=1}^k\Big(\frac{Z'}{Z}\Big(-\frac{2\pi L_\ell - i2\pi\tau_\ell}{N}\Big) + \overline{\frac{Z'}{Z}\Big(-\frac{2\pi L_\ell - i2\pi\tau_\ell}{N}\Big)}\Big)\,du.$$

Using Poisson summation as before in (30)

$$\begin{aligned}&\frac{1}{N}\Big(\frac{Z'}{Z}\Big(-\frac{2\pi L - i2\pi\tau}{N}\Big) + \overline{\frac{Z'}{Z}\Big(-\frac{2\pi L - i2\pi\tau}{N}\Big)}\Big)\\ &= \Big(\sum_{j=1}^N\sum_{r\in\mathbb{Z}}\frac{1}{N}\exp\Big(-2\pi L\tfrac{|r|}{N} + i2\pi\tau\tfrac{r}{N}\Big)e^{i2\pi r\theta_j}\Big) - 1\\ &= \Big(\sum_{j=1}^N\sum_{\nu\in\mathbb{Z}} h_{L,\tau}\big(N(\theta_j+\nu)\big)\Big) - 1,\end{aligned}$$

so

$$\begin{aligned}Q &= \lim_{N\to\infty}\mathop{\mathbf{E}}_{\mathcal{S}'_N}\prod_{\ell=1}^k\Big(\sum_i h_{L_\ell,\tau_\ell}(x_i) - 1\Big)\\ &= \mathop{\mathbf{E}}_{\mathcal{S}}\prod_{\ell=1}^k\Big(\sum_i h_{L_\ell,\tau_\ell}(x_i) - 1\Big).\end{aligned}$$

By Stirling's formula,

$$\begin{aligned}\tilde{G}_T(h_{L_\ell,\tau_\ell},t) &= \sum_\gamma h_{L,\tau}\big(\tfrac{\log T}{2\pi}(\gamma-t)\big) - \frac{\log t}{\log T}\int_{\mathbb{R}} h_{L,\tau}(x)\,dx + O_{L,\tau}\Big(\frac{1}{\log T}\Big)\\ &= \sum_\gamma h_{L,\tau}\big(\tfrac{\log T}{2\pi}(\gamma-t)\big) - 1 + O_{L,\tau}\Big(\frac{1}{\log T}\Big).\end{aligned}$$

Using Corollary 9.3, we thus have

$$\lim_{T\to\infty}\mathop{\mathbf{E}}_{Z_T}\prod_{\ell=1}^k\Big(\sum_i h_{L_\ell,\tau_\ell}(\xi_i)-1\Big) = \mathop{\mathbf{E}}_{\mathcal{S}}\prod_{\ell=1}^k\Big(\sum_i h_{L_\ell,\tau_\ell}(x_i)-1\Big),$$

for all k and all sets of positive constants $L_1,...,L_k$, and real constants $\tau_1,...,\tau_k$. Inductively removing lower order correlations from the above sums, we obtain for any such series of constants that

$$\mathop{\mathbf{E}}_{Z_T}\sum_{\substack{j_1,\ldots,j_k\\ \text{distinct}}} h_{L_1,\tau_1}(\xi_{j_1})\cdots h_{L_k,\tau_k}(\xi_{j_k}) = \mathop{\mathbf{E}}_{\mathcal{S}}\sum_{\substack{j_1,\ldots,j_k\\ \text{distinct}}} h_{L_1,\tau_1}(x_{j_1})\cdots h_{L_k,\tau_k}(x_{j_k}) + o(1) \tag{77}$$

But it is clear that if $\mathcal{A}_k := \mathrm{span}\{h_{L_1,\tau_1}\otimes\cdots\otimes h_{L_k,\tau_k}:\ L_1,...,L_k>0,\ \tau_1,...,\tau_k\in\mathbb{R}\}$, then $\mathcal{A}_k$ satisfies the conditions of Proposition 9.4, so that (77) implies the GUE Conjecture. This proves the theorem. □

13. Counts of almost primes

13.1. We turn at last to the proof of Theorem 2.5. It is easy to give a heuristic outline of the main ideas involved, although the rigorous proof that follows will entail substantial modifications.

We note that if $d\mathcal{P}(x)$ is the measure given by $d\psi(e^x)$, then it is easy to verify that

$$d\mathcal{P}*d\mathcal{P}(x) + x\,d\mathcal{P}(x) = d\psi_2(e^x)$$

In the same way, preceding entirely formally, if we define

$$dz_2(x) = dz*dz(x) + x\,dz(x),$$

this measure is given by the above measure $d\psi_2(e^x)$ minus a regular approximation:

$$dz_2(x) = d\tilde{\psi}_2(e^x),$$

where, recall, $\tilde{\psi}_2$ was defined in section 2 in equation (22). If we have proved Theorem 2.4 for more general f,g than separable functions, we could say that

$$\lim_{T\to\infty}\Psi_T^{2,1}(f\,;\,g) = \lim_{N\to\infty}\Theta_N^{2,1}(f\,;\,g), \tag{78}$$

where for $\beta>0$,

$$f(x_1,x_2) := \mathbf{1}_{[0,\beta)}(x_1+x_2)$$

$$g(y) := \mathbf{1}_{[0,\beta)}(y).$$

The advantage of this particular choice of f is that it allows us to convolve in the variables x_1 and x_2, and the left hand side of (78) reduces to

$$\frac{1}{\log^3 T}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\upsilon_T(x,y)\,dz*dz(x)\,dz(y),$$

while the right hand side reduces to

$$\frac{1}{N^3}\sum_{r,s\in\mathbb{N}_+}\mathbf{1}_{[0,\beta)}\Big(\tfrac{r}{N}\Big)\mathbf{1}_{[0,\beta)}\Big(\tfrac{s}{N}\Big)\int_{\mathcal{U}(N)}\Big(\sum_{r_1=1}^{r-1}[-\mathrm{Tr}(u^{r-r_1})][-\mathrm{Tr}(u^{r_1})]\Big)\overline{[-Tr(u^s)]}\,du$$
$$= -\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\delta(x-y)(x-1)_+\,dx\,dy + o(1)$$

by explicit computation with correlation functions. (cf. Theorem 11.2).

On the other hand, setting

$$f_1(x) := x\mathbf{1}_{[0,\beta)}(x),$$

$$g_1(y) := \mathbf{1}_{[0,\beta)}(y)$$

in the identity

$$\lim_{T\to\infty}\Psi_T^{1,1}(f_1\,;\,g_1) = \lim_{N\to\infty}\Theta_N^{1,1}(f_1\,;\,g_1)$$

we obtain

$$\lim_{T\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\upsilon_T(x,y)\,x\,dz(x)\,dz(y)$$
$$= \lim_{N\to]\infty}\sum_{r,s\in\mathbb{N}_+}\mathbf{1}_{[0,\beta)}\Big(\tfrac{r}{N}\Big)\mathbf{1}_{[0,\beta)}\Big(\tfrac{s}{N}\Big)\int_{\mathcal{U}(N)}[-r\mathrm{Tr}(u^r)]\overline{[-\mathrm{Tr}(u^s)]}\,du.$$

This left hand limit as $N\to\infty$ tends to

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\delta(x,y)\,x(x\wedge 1)\,dx\,dy.$$

By adding the results, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\upsilon_T(x,y)\,dz_2(x)\,dz(y)\sim\log^3 T\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)(x\wedge 1)^2\,dx.$$

The right hand side above can also be written in the form

$$\log^3 T\bigg(\lim_{N\to\infty}\frac{1}{N^3}\sum_{r,s}\mathbf{1}_{[0,\beta)}\Big(\tfrac{r}{N}\Big)\mathbf{1}_{[0,\beta)}\Big(\tfrac{s}{N}\Big)\int_{\mathcal{U}(N)}H_2(r)\overline{H_1(s)}\,du\bigg),$$

where for a unitary matrix $u\in\mathcal{U}(N)$ we define the quantities $H_j(r)$ inductively as follows: for $r\geq 1$,

$$H_1(r) := -\mathrm{Tr}(u^r) \tag{79}$$

$$H_j(r) := \sum_{s=1}^{r-1}H_{j-1}(r-s)H_1(s) + rH_{j-1}(r). \tag{80}$$

The similarity to the inductive definition (19) of the higher von Mangoldt functions should be clear.

We can generalize this argument. Letting $dz_j(x) := d\tilde{\psi}_j(x)$, we obtain

$$\lim_{T\to\infty}\frac{1}{\log^{j+k}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{[0,\beta)}(x)\mathbf{1}_{[0,\beta)}(y)\upsilon_T(x,y)\,dz_j(x)\,dz_k(y)$$
$$= \lim_{N\to\infty}\sum_{r,s\in\mathbb{N}_+}\mathbf{1}_{[0,\beta)}\Big(\tfrac{r}{N}\Big)\mathbf{1}_{[0,\beta)}\Big(\tfrac{s}{N}\Big)\int_{\mathcal{U}(N)}H_j(r)\overline{H_k(s)}\,du.$$

It is from Lemma 13.1 that we can simplify the random matrix part of this identity. On the other hand, as in section 1, the arithmetic side is given by

$$\lim_{T\to\infty} \frac{T}{\log^{j+k} T} \int_1^{T^\beta} \Big(\tilde{\psi}_j(\tau + \tfrac{\tau}{T}) - \tilde{\psi}_j(\tau)\Big)\Big(\tilde{\psi}_k(\tau + \tfrac{\tau}{T}) - \tilde{\psi}_k(\tau)\Big) \frac{d\tau}{\tau^2}.$$

In this manner we have arrived at a (purely formal) derivation of Theorem 23. We are prevented from making this argument rigorous in the above form in that we have proved Theorem 2.4 only for functions f, g that are separable. In particular, we cannot approximate $f(x_1, x_2) = \mathbf{1}_{[0,\beta)}(x_1 + x_2)$ with a single separable function. Even to approximate this function with a linear combination of separable functions will not do, as we have proved no continuity properties for Ψ_T (an integral against signed measures) in the $T \to \infty$ limit. Equation (78) is therefore unjustified for the test functions we have made use of. We are therefore left with two routes to make the above sketch rigorous. In the first we could reprove Theorem 2.4 for test functions f and g that are not separable. This should certainly be possible, but will entail making the proof of the theorem more complicated. (The reader is encouraged to try to come up with a simple argument!) In the second possible approach, we make use of seperable functions that allow for convolution – these are exactly the exponential functions, and therefore the case we have considered in Theorems 2.2 and 12.1. This is the route we shall take. It involves the additional complication that exponential functions are not compactly supported, and this fact entails a sort of gymnastics that we must go through in the proof that follows.

13.2.

Proof of Theorem 2.5. We note that Theorem 12.1 may be rewritten in the form that, conditioned on the GUE Conjecture

$$\lim_{T\to\infty} \frac{1}{\log^{j+k} T} \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{j} P_\ell\Big(\frac{d}{dA_\ell}\Big)\Big(\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A_\ell}{\log T} + it\Big)\Big) \overline{\prod_{\ell'=1}^{k} Q_{\ell'}\Big(\frac{d}{dB_{\ell'}}\Big)\Big(\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \tfrac{B_{\ell'}}{\log T} + it\Big)\Big)}\, dt$$

$$= \lim_{N\to\infty} \frac{1}{N^{j+k}} \int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} P_\ell\Big(\frac{d}{dA_\ell}\Big)\Big(\frac{Z'}{Z}\Big(\tfrac{A_\ell}{N}\Big)\Big) \overline{\prod_{\ell'=1}^{k} Q_{\ell'}\Big(\frac{d}{dB_{\ell'}}\Big)\Big(\frac{Z'}{Z}\Big(\tfrac{B_{\ell'}}{N}\Big)\Big)}\, du,$$

for any polynomials $P_1, .., P_j, Q_1, ..., Q_k$, where $\sigma_1(t) := \frac{1}{2\pi}\Big(\frac{\sin t/2}{t/2}\Big)^2$ as in (50). We will use this definition of σ_1 throughout this proof.

Because

$$\frac{\zeta^{(j)}}{\zeta}(s) = \Big(\frac{\zeta'}{\zeta} + \frac{d}{ds}\Big)\frac{\zeta^{(j-1)}}{\zeta}(s),$$

and likewise for $Z^{(j)}/Z$, we can inductively show from Theorem 12.1,

$$\lim_{T\to\infty} \frac{1}{\log^{J+K} T} \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \prod_{\ell=1}^{J} P_\ell\Big(\frac{d}{dA_\ell}\Big)\Big(\frac{\zeta^{(j_\ell)}}{\zeta}\Big(\tfrac{1}{2} + \tfrac{A_\ell}{\log T} + it\Big)\Big)$$
$$\times \overline{\prod_{\ell'=1}^{K} Q_{\ell'}\Big(\frac{d}{dB_{\ell'}}\Big)\Big(\frac{\zeta^{(k_{\ell'})}}{\zeta}\Big(\tfrac{1}{2} + \tfrac{B_{\ell'}}{\log T} + it\Big)\Big)}\, dt$$

$$= \lim_{N\to\infty} \frac{1}{N^{J+K}} \int_{\mathcal{U}(N)} \prod_{\ell=1}^{J} P_\ell\Big(\frac{d}{dA_\ell}\Big)\Big(\frac{Z^{(j_\ell)}}{Z}\Big(\tfrac{A_\ell}{N}\Big)\Big) \overline{\prod_{\ell'=1}^{k} Q_{\ell'}\Big(\frac{d}{dB_{\ell'}}\Big)\Big(\frac{Z^{(k_{\ell'})}}{Z}\Big(\tfrac{B_{\ell'}}{N}\Big)\Big)}\, du,$$

We specialize to the case $J = K = 1$ and $A_1 = B_1$ real to obtain

$$\lim_{T\to\infty} \frac{1}{\log^{j+k} T} \int_{\mathbb{R}} \frac{\sigma_1(t/T)}{T} \left((-1)^j \frac{\zeta^{(j)}}{\zeta}\left(\tfrac{1}{2} + \tfrac{A}{\log T} + it\right)\right)\left((-1)^k \frac{\zeta^{(k)}}{\zeta}\left(\tfrac{1}{2} + \tfrac{A}{\log T} - it\right)\right) dt$$
$$= \lim_{N\to\infty} \frac{1}{N^{j+k}} \int_{\mathcal{U}(N)} \left((-1)^j \frac{Z^{(j)}}{Z}\left(\tfrac{A}{N}\right)\right)\left((-1)^k \frac{Z^{(k)}}{Z}\left(\tfrac{A}{N}\right)\right) du. \quad (81)$$

This is in fact the identity we need, albeit in a somewhat vieled form. We now prove the theorem in four steps. In the first two steps, our development mimics the elegant approach in [**26**], which in turn draws from Selberg [**56**].

Step 1: We show for positive A and

$$f_\kappa(s) := \frac{e^{\kappa s} - 1}{s}$$

that for $\alpha := \frac{1}{2} + \frac{A}{\log T}$,

$$\int_0^\infty \frac{1}{r^{2\alpha}} \left(\tilde{\psi}_j(e^\kappa r) - \tilde{\psi}_j(r)\right)\left(\tilde{\psi}_k(e^\kappa r) - \tilde{\psi}_k(r)\right) dr \quad (82)$$
$$= \int_{\mathbb{R}} \left((-1)^j \frac{\zeta^{(j)}}{\zeta}(\alpha + it)\right)\left((-1)^k \frac{\zeta^{(k)}}{\zeta}(\alpha - it)\right) \frac{|f_\kappa(\alpha+it)|^2}{2\pi}\, dt.$$

Step 2: We show that for κ_1 such that $e^{\kappa_1} - 1 = 1/T$, and α and f defined as in step 1,

$$\int_{\mathbb{R}} \left((-1)^j \frac{\zeta^{(j)}}{\zeta}(\alpha + it)\right)\left((-1)^k \frac{\zeta^{(k)}}{\zeta}(\alpha - it)\right) \frac{|f_\kappa(\alpha+it)|^2}{2\pi}\, dt$$
$$- \frac{1}{T} \int_{\mathbb{R}} \left((-1)^j \frac{\zeta^{(j)}}{\zeta}(\alpha + it)\right)\left((-1)^k \frac{\zeta^{(k)}}{\zeta}(\alpha - it)\right) \frac{\sigma(t/T)}{T}\, dt \quad (83)$$
$$= O_A\Big(\frac{\log^{2(j+k)+1} T}{T}\Big),$$

Step 3: We combine these steps with (81) and the random matrix statistic Lemma 13.1. We obtain that for any positive constant A

$$\int_0^\infty \frac{1}{r^{2+2A/\log T}} \tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, dr \sim \frac{\log^{j+k} T}{T} \frac{jk}{j+k-1} \int_0^\infty e^{-2Ay}(y\wedge 1)^{j+k-1}\, dy. \quad (84)$$

Step 4: We use a Tauberian argument to pass between the weights $e^{-\beta x}$ and $\mathbf{1}_{[0,\beta)}(x)$, thereby showing that (84) implies the covariance asymptotic (23) for any constant $\beta > 0$.

Having verified these steps, our proof will be complete.

Step 1: It follows from a standard argument in residue calculus (using the bound of Appendix A for $\zeta^{(j)}/\zeta$ at large heights) that when $x > 0$ is not an integer, for $\alpha \in (1/2, 1)$,

$$\tilde{\psi}_j(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left((-1)^j \frac{\zeta^{(j)}}{\zeta}(s)\right) \frac{x^s}{s}\, ds.$$

Continuing the mimic the arguments [**26**], differencing the values when $x = e^{\tau+\kappa}$ and $x = e^\tau$ gives for almost all τ,

$$\frac{\tilde{\psi}_j(e^\kappa e^\tau) - \tilde{\psi}_j(e^\tau)}{e^{\tau\alpha}} = \frac{1}{2\pi} \int_{-\infty}^\infty (-1)^j \frac{\zeta^{(j)}}{\zeta}(\alpha + it)\Big(\frac{e^{\kappa(\alpha+it)} - 1}{\alpha + it}\Big) e^{it\tau}\, dt.$$

The right hand side is the inverse Fourier transform of $(-1)^j\zeta^{(j)}/\zeta(\alpha+i2\pi t)f_\kappa(\alpha+i2\pi t)$, while the left hand side is obviously real valued. It is moreover easy to see from the elementary estimates in Appendix A that the left hand side is square integrable in τ and so by an application of Plancherel

$$\int_{\mathbb{R}} \frac{\big(\tilde\psi_j(e^\kappa e^\tau)-\tilde\psi_j(e^\tau)\big)\big(\tilde\psi_k(e^\kappa e^\tau)-\tilde\psi_k(e^\tau)\big)}{e^{2\tau\alpha}}\,d\tau$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}\bigg((-1)^j\frac{\zeta^{(j)}}{\zeta}(\alpha+it)\bigg)\bigg((-1)^k\frac{\zeta^{(k)}}{\zeta}(\alpha-it)\bigg)|f_\kappa(\alpha+it)|^2\,dt.$$

Making the change of variables $r=e^\tau$ and setting $\alpha=1/2+A/\log T$, this is (82).

Step 2: We quote the estimate from [**26**], that for $\alpha\le 1$, $0<\kappa\le 1$,

$$|f_\kappa(\alpha+it)|^2-|f_\kappa(it)|^2=O\Big(\tfrac{\kappa}{|t|^2}\,\wedge\,\kappa^2\Big).$$

Likewise, because $|f_\kappa(it)|^2=\Big(\frac{\sin\kappa t/2}{t/2}\Big)^2$ and

$$\sin^2 x-\sin^2 y=O\big(|x-y|\,\wedge\,1\big),$$

we have for real κ_1,κ_2 and $t\ge 1$

$$|f_{\kappa_1}(it)|^2-|f_{\kappa_2}(it)|^2=O\Big(\tfrac{|\kappa_1-\kappa_2|}{|t|}\,\wedge\,\tfrac{1}{|t|^2}\Big),$$

while for $t\le 1$ and $\kappa\le 1$, clearly

$$|f_\kappa(it)|^2=O(\kappa^2).$$

We also make use of the basic pointwise bound proved in Appendix A,

$$\frac{\zeta^{(j)}}{\zeta}(\alpha+it)=O\Big(\frac{\log^j(|t|+2)}{(\alpha-1/2)^j}\Big),$$

for $|\alpha+it-1|\ge 1/4$, say.

We let κ_1 be such that $e^{\kappa_1}-1=1/T$ and $\kappa_2=1/T$. Note that $f_{\kappa_2}(it)=\sigma_1(t/T)/T^2$, and $\kappa_1-\kappa_2=O(1/T^2)$. Hence, the left hand side of (83) has the bound

$$\lesssim\int_{|t|\ge 1}\frac{\log^{j+k}(|t|+2)}{A^{j+k}/\log^{j+k}T}\Big(O\Big(\tfrac{1}{T|t^2|}\,\wedge\,\tfrac{1}{T^3}\Big)+O\Big(\tfrac{1}{T^2|t|}\,\wedge\,\tfrac{1}{|t|^2}\Big)\Big)\,dt+\int_{|t|<1}O\Big(\tfrac{1}{T^2}\Big)\,dt$$
$$\lesssim_A \log^{j+k}T\bigg(\int_1^T\frac{1}{T^3}+\int_T^\infty\frac{1}{T|t|^3}+\int_1^{T^2}\frac{1}{T^2|t|}+\int_{T^2}^\infty\frac{1}{|t|^2}\,dt\bigg)+O\Big(\tfrac{1}{T^2}\Big)$$
$$\lesssim_A \frac{\log^{2(j+k)+1}T}{T^2},$$

as claimed.

Step 3: We turn to evaluating the right hand side of (81).

Recall that the random matrix quantities $H_j(r)$ are defined by (79) and (80). We demonstrate inductively that

$$(-1)^j\frac{Z^{(j)}}{Z}(\beta)=\sum_{r=1}^\infty e^{-\beta r}H_j(r). \qquad (85)$$

For, the identity (7) says just that

$$-\frac{Z'}{Z}(\beta) = \sum_{r=1}^{\infty} e^{-\beta r} H_1(r),$$

while the fact that

$$(-1)^j \frac{Z^{(j)}}{Z}(\beta) = \Big(- \frac{Z'}{Z}(\beta) - \frac{d}{d\beta}\Big) \Big((-1)^{j-1} \frac{Z^{(j-1)}}{Z}(\beta)\Big)$$

and the definition (79) and (80) of $H_j(r)$ completes the induction to $j > 1$.

We cite the following result, proved as Lemma 2.2 in [**54**],

Lemma 13.1.

$$\int_{\mathcal{U}(N)} H_j(r)\overline{H_k(r)}\, du = \delta_{rs} \sum_{\nu=1}^{r\wedge N} \big(\nu^j - (\nu-1)^j\big)\big(\nu^k - (\nu-1)^k\big).$$

Note that the well-known identity (17) is the case $j = k = 1$. See the appendix of [**54**] for further commentary on the connection of this identity to other results in random matrix theory.

From Lemma 13.1 therefore,

$$\int_{\mathcal{U}(N)} \Big((-1)^j \frac{Z^{(j)}}{Z}(\beta)\Big)\Big((-1)^k \frac{Z^{(k)}}{Z}(\beta)\Big)\, du = \sum_{r=1}^{\infty} e^{-2\beta r} \sum_{\nu=1}^{r\wedge N} (\nu^j - (\nu-1)^j)(\nu^k - (\nu-1)^k).$$

(The interchange of integration and summation is easy to justify, as for fixed N, $H_j(r)$ is bounded.) Hence,

$$\begin{aligned} &\frac{1}{N^{j+k}} \int_{\mathcal{U}(N)} \Big((-1)^j \frac{Z^{(j)}}{Z}\Big(\frac{A}{N}\Big)\Big)\Big((-1)^k \frac{Z^{(k)}}{Z}\Big(\frac{A}{N}\Big)\Big) \\ &= \frac{1}{N} \sum_{r=1}^{\infty} e^{-2Ar/N} \frac{1}{N^{j+k-1}} \Big(\tfrac{jk}{j+k-1} (r\wedge N)^{j+k-1} + O((r\wedge N)^{j+k-2})\Big) \\ &\sim \tfrac{jk}{j+k-1} \int_0^{\infty} e^{-2Ay} (y \,\wedge\, 1)^{j+k-1}\, dy, \end{aligned}$$

since the sum over r is just a Riemann sum.

Using (82), (83), and (81) in succession, we arrive at (84).

Step 4: In the first place note that

$$\int_0^1 \frac{1}{r^{2+2A/\log T}} \tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, dr = O\bigg(\int_0^1 \frac{1}{r^{2A/\log T}} \frac{|\log r|^{j+k-2}}{T}\, dr\bigg) = O\Big(\frac{\log^{j+k-1} T}{T}\Big),$$

so (84) is equivalent to

$$\int_1^{\infty} \frac{1}{r^{2+2A/\log T}} \tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, dr \sim \frac{\log^{j+k} T}{T} \frac{jk}{j+k-1} \int_0^{\infty} e^{-2Ay}\, (y \,\wedge\, 1)^{j+k-1}\, dy. \tag{86}$$

With this simplification, we move on to the Tauberian part of the proof, that (86) implies (23).

Note that for any continuous function ϕ of compact support, and for any $\epsilon > 0$, there is a polynomial P so that

$$\big|P\big(\tfrac{1}{w}\big) - \phi(w)\big| \leq \epsilon/w \qquad \text{for } w \geq 1. \tag{87}$$

For, note that by its compact support, $\phi(1/x)/x$ is continuous on the interval $[0,1]$ (defined by continuity to take the value 0 at $x = 0$). Hence by Weierstrass's approximation theorem, there is some polynomial Q so that

$$|Q(x) - \phi(1/x)/x| < \epsilon \qquad \text{for all } x \in [0,1].$$

$P(x) := xQ(x)$ thus satisfies (87).

We use this to show that for any continuous f of compact support,

$$\int_1^\infty f\Big(\tfrac{\log r}{\log T}\Big)\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, dr \sim \frac{\log^{j+k} T}{T}\frac{jk}{j+k-1}\int_0^\infty f(y)(y \,\wedge\, 1)^{j+k-1}\, dy. \tag{88}$$

For if $\phi(x) = f(\log x)$, then for P as in (87), using Cauchy-Schwarz,

$$\begin{aligned}
&\int_1^\infty \Big(f\Big(\tfrac{\log r}{\log T}\Big) - P\Big(\frac{1}{r^{1/\log T}}\Big)\Big)\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, \frac{dr}{r^2} \\
&\leq \Big(\int_1^\infty \frac{\epsilon}{r^{1/\log T}}\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)^2\,\frac{dr}{r^2}\Big)^{1/2}\Big(\int_1^\infty \frac{\epsilon}{r^{1/\log T}}\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)^2\,\frac{dr}{r^2}\Big)^{1/2} \\
&\lesssim_{j,k} \frac{\log^{j+k} T}{T}\epsilon.
\end{aligned}$$

by an applications of (86) in the case $j = k$.

On the other hand, from (86) again,

$$\begin{aligned}
&\int_1^\infty P\Big(\frac{1}{r^{1/\log T}}\Big)\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, \frac{dr}{r^2} \\
&= \frac{\log^{j+k} T}{T}\frac{jk}{j+k-1}\Big(\int_0^\infty P(e^{-y})\,(y \,\wedge\, 1)^{j+k-1}\, dy + o(1)\Big) \\
&= \frac{\log^{j+k} T}{T}\frac{jk}{j+k-1}\Big(\int_0^\infty f(y)\,(y \,\wedge\, 1)^{j+k-1}\, dy + O(\epsilon) + o(1)\Big).
\end{aligned}$$

As ϵ was arbitrary, this proves (88) for continuous and compactly supported f.

We want finally to show that (88) remains true when $f = \mathbf{1}_{[0,\beta)}$. This function is not continuous, but for any $\epsilon > 0$, plainly there exist continuous functions of compact support, f_1 and h, so that

$$f(x) = f_1(x) \text{ for } x \in [0,\beta)$$

$$|f(x) - f_1(x)| \leq h(x) \text{ for all } x, \text{ and } \int_0^\infty h(x)\, dx < \epsilon.$$

Hence,

$$\begin{aligned}
&\int_1^\infty \Big(f\Big(\tfrac{\log r}{\log T}\Big) - f_1\Big(\tfrac{\log r}{\log T}\Big)\Big)\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)\, \frac{dr}{r^2} \\
&\leq \Big(\int_1^\infty h\Big(\tfrac{\log r}{\log T}\Big)\tilde{\psi}_j\big(r;\, \tfrac{r}{T}\big)^2\,\frac{dr}{r^2}\Big)^{1/2}\Big(\int_1^\infty h\Big(\tfrac{\log r}{\log T}\Big)\tilde{\psi}_k\big(r;\, \tfrac{r}{T}\big)^2\,\frac{dr}{r^2}\Big)^{1/2} \\
&\lesssim_{j,k} \frac{\log^{j+k} T}{T}\epsilon.
\end{aligned}$$

In the second line we used the positivity of $\tilde{\psi}_j^2$ and $\tilde{\psi}_k^2$ to replace $|f - f_1|$ by its majorant h.

Clearly

$$\left| \int_0^\infty \big(f(y) - f_1(y)\big)\,(y \wedge 1)^{j+k-1}\,dy \right| < \epsilon$$

as well. Because ϵ is arbitrary, this proves that (88) is true even when $f = \mathbf{1}_{[0,\beta)}$, which is what we sought to show.

This completes step 4, and therefore the proof of Theorem 2.5. □

13.3. It is natural to ask whether weights more general than Λ_j on almost primes can be defined such that an estimate like that in Theorem 2.5 will imply the GUE Conjecture. Such a class of weights can indeed be defined, but the resulting asymptotic relation cannot be simple. We will describe such a result here but only outline the proof.

For a vector $\mathbf{a} = (a_1, ..., a_j)$ with positive integer entries, define the function

$$\Lambda_{[\mathbf{a}]}(n) = \sum_{n_1 \cdots n_j = n} (\log n)^{a_1 - 1}\Lambda(n_1) \cdots (\log n)^{a_j-1}\Lambda(n_j),$$

supported on almost primes and with Dirichlet series

$$\sum_n \frac{\Lambda_{[\mathbf{a}]}(n)}{n^s} = \prod_{\ell=1}^{j} (-1)^{a_j - 1}\Big(\frac{\zeta'}{\zeta}\Big)^{a_\ell - 1}(s).$$

If

$$\psi_{[\mathbf{a}]}(x) = \sum_{n \le x} \Lambda_{[\mathbf{a}]}(n),$$

then it may be seen that $\psi_{[\mathbf{a}]}(x) = xQ_{\mathbf{a}}(\log x) + o(x)$, where $Q_{\mathbf{a}}(\log x)$ is a degree $|\mathbf{a}|$ polynomial, where we follow the convention that

$$|\mathbf{a}| = a_1 + \cdots + a_j. \tag{89}$$

Define $\widetilde{\psi}_{[\mathbf{a}]}(x) - xQ_{\mathbf{a}}(x)$, and

$$\widetilde{\psi}_{[\mathbf{a}]}(x; H) = \widetilde{\psi}_{[\mathbf{a}]}(x + H) - \widetilde{\psi}_{[\mathbf{a}]}(x).$$

Furthermore define a random matrix analogue of this almost prime weight: for $r \ge 1$, define

$$T_{[\mathbf{a}]}(r) = \sum_{\substack{r_1 + \cdots + r_j = r \\ r_i \ge 1}} \prod_{\ell=1}^{j} r_\ell^{a_\ell - 1}(-\mathrm{Tr}(u^{r_\ell})).$$

This function has the generating series

$$\sum_{r=1}^{\infty} T_{[\mathbf{a}]}(r) e^{-\beta r} = \prod_{\ell=1}^{j} (-1)^{a_\ell - 1}\Big(\frac{Z'}{Z}\Big)^{(a_\ell - 1)}(\beta).$$

THEOREM 13.2. *(On RH.) The GUE Conjecture is equivalent to the following statement: for all fixed $j, k \ge 1$, fixed vectors $\mathbf{a} \in \mathbb{N}_+^j$, $\mathbf{b} \in \mathbb{N}_+^k$ and fixed $\beta > 0$,*

$$\int_1^X \widetilde{\psi}_{[\mathbf{a}]}(x;\delta x)\widetilde{\psi}_{[\mathbf{b}]}(x;\delta x)\,\frac{dx}{x^2} = \frac{(\log T)^{|\mathbf{a}|+|\mathbf{b}|}}{T}\Big(W_{\mathbf{a},\mathbf{b}}(\beta) + o(1)\Big),$$

where $X = T^\beta$ and $\delta = 1/T$, and

$$W_{\mathbf{a},\mathbf{b}}(\beta) = \lim_{N\to\infty} \frac{1}{N^{|\mathbf{a}|+|\mathbf{b}|}} \sum_{r \le \beta N} \int_{\mathcal{U}(N)} T_{[\mathbf{a}]}(r)\overline{T_{[\mathbf{b}]}(r)}\,du.$$

As in other results of this sort in this paper it is not immediately evident that the limit defining $W_{\mathbf{a},\mathbf{b}}(\beta)$ exists, but this is relatively easy to show.

Theorem 13.2 may be proved using the method above for proving Theorem 2.5, relying upon the following result:

THEOREM 13.3. *(On RH.) The GUE Conjecture is equivalent to the statement that for all fixed $j,k \geq 1$, fixed vectors $\mathbf{a} = (a_1, ..., a_j) \in \mathbb{N}_+^j$, $\mathbf{b} = (b_1, ..., b_j) \in \mathbb{N}_+^k$ and a fixed constant A with positive real part,*

$$\lim_{T\to\infty} \frac{1}{\log^{|\mathbf{a}|+|\mathbf{b}|} T}\Big(\frac{1}{T}\int_T^{2T} \prod_{\ell=1}^{j} \Big(\frac{\zeta'}{\zeta}\Big)^{(a_\ell - 1)}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big) \overline{\prod_{\ell'=1}^{j} \Big(\frac{\zeta'}{\zeta}\Big)^{(b_{\ell'} - 1)}\Big(\tfrac{1}{2} + \tfrac{A}{\log T} + it\Big)}\, dt\Big) \tag{90}$$

exists and is equal to

$$\lim_{N\to\infty} \frac{1}{N^{|\mathbf{a}|+|\mathbf{b}|}}\Big(\int_{\mathcal{U}(N)} \prod_{\ell=1}^{j} \Big(\frac{Z'}{Z}\Big)^{(a_\ell - 1)}\Big(\frac{A}{N}\Big) \overline{\prod_{\ell'=1}^{j} \Big(\frac{Z'}{Z}\Big)^{(b_{\ell'} - 1)}\Big(\frac{A}{N}\Big)}\, du\Big). \tag{91}$$

Here $f^{(n)}$ denotes the nth derivative of a function and $|\mathbf{a}| = a_1 + \cdots + a_j$ and $|\mathbf{b}| = b_1 + \cdots b_k$.

Note that in contrast to Theorem 2.2, Theorem 13.3 requires only a single shift A (but also arbitrarily many differentiations).

Theorem 13.3 in turn is proved in much the same way as Theorem 2.2 and we say only a few words about the proof here. Note that derivatives of the logarithmic derivative of the zeta function or a characteristic polynomial are characterized by the relations (73) and (74). Thus in the same way that Theorem 2.2 depends upon the fact that linear combinations of functions $\exp(-A_1x_1 - \cdots - A_jx_j)$ are dense in $C_c(\mathbb{R}_+^j)$, Theorem 13.3 depends upon nearly the same ideas and the fact that linear combinations of functions $x_1^{a_1-1} \cdots x_j^{a_j-1} \exp(-A(x_1 + \cdots + x_j))$ are also dense in $C_c(\mathbb{R}_+^j)$.

Appendix A. Counts of almost primes in long intervals

We made us of the following estimates earlier; as in the rest of the document, we require the Riemann hypothesis for their proof.

THEOREM A.1. *(On RH.) For fixed j with $\sigma \in (1/2, 1)$ and $|\sigma + it - 1| \geq 1/4$*

$$\frac{\zeta^{(j)}}{\zeta}\Big(\sigma + it\Big) = O\Big(\frac{\log^j(|t| + 2)}{(\sigma - 1/2)^j}\Big).$$

The region above are chosen so that they do not include the singularity at $\sigma + it = 1$.

THEOREM A.2. *(On RH.) For fixed j,*

$$\psi_j(x) = \int_0^x j \log^{j-1} y\, dy + O_j\big(x^{1/2} \log^{2j+1} x\big).$$

We prove Theorem A.2 on the basis of Theorem A.1. The error term bound $O(x^{1/2} \log^{2j+1} x)$ is not optimal; by refining our technique (by using the mean value estimates in this paper for instance), one can obtain an error term of $O(x^{1/2} \log^{j+1} x)$, an estimate on the level of the classical von Koch estimate for $j = 1$. It is likely that even this estimate is not optimal (for

$j=1$ for instance Montgomery has conjectured the error term is of order $x^{1/2}(\log\log\log x)^2)$ but either estimate will be sufficient for our purposes.

Proof of Theorem A.1. We have for $|t|\geq 1$ and $\sigma > 1/2$

$$\begin{aligned}\frac{\zeta'}{\zeta}\Big(\sigma+it\Big) &= \sum_{|\gamma-t|\leq 1}\frac{1}{\sigma+it-(1/2+i\gamma)} + O\big(\log(|t|+2)\big)\\ &= O\Big(\frac{\log(|t|+2)}{\sigma-1/2}\Big)+O\big(\log(|t|+2)\big),\end{aligned}$$

with the first line following from Lemma 12.1 of [**48**] (essentially taking a logarithmic derivative of a Hadamard product), and the second from bounding the number of zeros that lie in a unit interval at height t. We show inductively that

$$\frac{\zeta^{(j)}}{\zeta}\Big(\sigma+it\Big) = O_j\big(\big((\sigma-1/2)^{-1}\vee 1\big)^j\log^j(|t|+2)\big)$$

for $|t|\geq 1$; we have just demonstrated it for $j=1$. Suppose we have the estimate for $\zeta^{(j-1)}/\zeta$. Then for $s=\sigma+it$, $|t|\geq 2$ and $\delta=(\sigma-1/2)^{-1}\wedge 1$,

$$\begin{aligned}\frac{\zeta^{(j)}}{\zeta}(s) &= \Big(\frac{\zeta^{(j-1)}}{\zeta}\Big)'(s)+\frac{\zeta'}{\zeta}(s)\frac{\zeta^{(j-1)}}{\zeta}(s)\\ &= \frac{1}{2\pi i}\int_{|z-s|=\delta}\frac{\zeta^{(j-1)}}{\zeta}(z)\frac{dz}{(z-s)^2}+O_j\big(\big((\sigma-1/2)^{-1}\vee 1\big)^j\log^j(|t|+2)\big)\\ &= O_j\big(\big((\sigma-1/2)^{-1}\vee 1\big)^j\log^j(|t|+2)\big).\end{aligned}$$

For $t\in(1,2)$ clearly

$$\frac{\zeta^{(j)}}{\zeta}(s) = O_j(1),$$

which completes our induction.

As moreover for $|t|\in(0,1)$ but $|\sigma+it-1|\geq 1/4$,

$$\frac{\zeta^{(j)}}{\zeta}\Big(\sigma+it\Big) = O_j(1),$$

we have proved the theorem. □

To prove Theorem A.2 we reference Lemma 3.12 from Titchmarsh's tract [**62**]:

Lemma A.3 (Lemma 3.12 of [**62**]). *Let*

$$f(s)=\sum_{n=1}^\infty\frac{a_n}{n^s}\quad(\sigma>1)$$

where $a_n = O(\rho(n))$, $\rho(n)$ non-decreasing, and

$$\sum_{n=1}^\infty\frac{|a_n|}{n^\sigma}=O\Big(\frac{1}{(\sigma-1)^\alpha}\Big),$$

as $\sigma \to 1$*. Then if* $c > 0$*,* $\sigma + c > 1$*,* x *not an integer, and* N *is the integer nearest to* x*,*

$$\sum_{n\leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw + O\Big(\frac{x^c}{T(\sigma+c-1)^\alpha}\Big) + O\Big(\frac{\rho(2x)x^{1-\sigma}\log x}{T}\Big) + O\Big(\frac{\rho(N)x^{1-\sigma}}{T|x-N|}\Big).$$

Proof of Theorem A.2. Using the lemma with $a_n = \Lambda_j(n)$, we have

$$f(s) = (-1)^j \frac{\zeta^{(j)}}{\zeta}(s),$$

$$a_n = O(\log^j n)$$

$$\sum_{n=1}^\infty \frac{|a_n|}{n^\sigma} = (-1)^j\frac{\zeta^{(j)}}{\zeta}(\sigma) \sim \frac{j!}{(\sigma-1)^j}.$$

Setting $s = 1/2$, $c = 3/4$, and $T = x^2$ for $x = N + 1/2$, we have

$$\begin{aligned}\sum_{n\leq x}\frac{\Lambda_j(n)}{\sqrt{n}} =& \frac{1}{2\pi i}\int_{3/4-iT}^{3/4+iT} (-1)^j\frac{\zeta^{(j)}}{\zeta}\big(\tfrac12 + w\big)\frac{x^w}{w}\,dw + o(1)\\ =& \operatorname*{Res}_{w=1/2}\Big(\frac{j!}{(w-1/2)^j}\frac{x^w}{w}\Big)\\ &+ \frac{1}{2\pi i}\bigg(\int_{1/\log T+iT}^{3/4+iT} + \int_{1/\log T-iT}^{1/\log T+iT} + \int_{3/4-iT}^{1/\log T-iT}\bigg)(-1)^j\frac{\zeta^{(j)}}{\zeta}\big(\tfrac12+w\big)\frac{x^w}{w}\,dw + o(1)\\ =& \int_0^x \frac{j\log^{j-1} y}{\sqrt{y}}\,dy + O\bigg(\int_{-T}^{T}\frac{\log^j T\log^j(|t|+2)}{|\frac{1}{\log T}+it|}\,dt\bigg) + o(1)\\ =& \int_0^x \frac{j\log^{j-1} y}{\sqrt{y}}\,dy + O(\log^{2j+1} x).\end{aligned}$$

Because $\log^{2j+1} x$ is a slowly growing function we obtain this for all x, not only $x = N+1/2$. The theorem then follows from partial integration. □

Appendix B. The sine-kernel determinantal point process

A point process $(X, \mathfrak{F}, \mathbb{P})$ on $\mathbb{R}$ is a probability measure $\mathbb{P}$ on the σ-algebra $(X, \mathfrak{F})$, where X is the set of locally finite configurations of sequences of real numbers:

$$X := \{\xi = ((..., \xi_{-1}, \xi_0, \xi_1, ...)) : \xi_i \in \mathbb{R}\ \forall i \in \mathbb{Z},\ \text{and for any compact } K \subset \mathbb{R},\ \#_K(\xi) = \#\{i : \xi_i \in K\} \leq \infty\}$$

and $\mathfrak{F}$ is the σ-algebra with a basis consisting of the cylinder sets

$$C_n^B := \{\xi \in X : \#_B(\xi) = n\}$$

where $n = 0, 1, 2, ...$ and B is any bounded Borel subset of $\mathbb{R}$. Further discussions of this definition can be found in [**35**, Ch. 16] or [**58**]. An account of point processes introduced from the perspective of zeta zeros can be found in [**12**].

For any Borel $B_1, ..., B_k$, the expectation

$$\mathbf{E}\sum_{j_1,...,j_k}\mathbf{1}_{B_1}(\xi_{j_1})\cdots\mathbf{1}_{B_k}(\xi_{j_k}) = \mathop{\mathbf{E}}_{(X,\mathfrak{F},\mathbb{P})}\#_{B_1}(\xi)\cdots\#_{B_k}(\xi)$$

can be evaluated (possibly as infinity) and, from approximation by simple functions, for any non-negative measurable $\eta\colon \mathbb{R}^k \to \mathbb{R}$, the expectation

$$\mathbf{E} \sum_{j_1,\ldots,j_k} \eta(\xi_{j_1},...,\xi_{j_k})$$

can be evaluated as well. Suppose these quantities are finite for all $\eta \in C_c(\mathbb{R}^k)$. Then by a combinatorial sieving procedure, so too can

$$\mathbf{E} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(\xi_{j_1},...,\xi_{j_k})$$

be evaluated. For instance,

$$\mathbf{E} \sum_{j_1 \neq j_2} \eta(\xi_{j_1},\xi_{j_2}) = \mathbf{E} \sum_{j_1,j_2} \eta(\xi_{j_1},\xi_{j_2}) - \mathbf{E} \sum_{j} \eta(\xi_j,\xi_j).$$

As long as the point process satisfies the mild condition that $\mathbf{E}\,\#_K(\xi)^k < \infty$ for all bounded intervals K, this defines a measure $d\mu_k$ on $\mathbb{R}^k$, called the k-level joint intensity measure,

$$\mathbf{E} \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(\xi_{j_1},...,\xi_{j_k}) = \int_{\mathbb{R}^k} \eta(x_1,...,x_k)\, d\mu_k(x_1,...,x_k).$$

The details of a proof of the existence of μ_k via Riesz representation can be found in e.g. [**44**, Prop 3.2] or [**43**, Thm. A.1]. These measures should be thought of as being analogues for point processes of moments of random variables.

By no means do all collections of measures $\{d\mu_1, d\mu_2, ...\}$ on $\mathbb{R}^1, \mathbb{R}^2, ...$ correspond to the joint intensity of a point process, but in the case that

$$d\mu_k(x_1,...,x_k) = \det_{k\times k}\big(K(x_i - x_j)\big)\, dx_1 dx_2 \cdots dx_k$$

it is known that there exists a unique point process, labeled 'the sine-kernel determinantal point process', with these joint intensities. Details of its construction and a more general account of the theory of determinantal point processes can be found in [**58**].

Appendix C. On different formulations of the GUE Conjecture

The GUE Conjecture is sometime formulated in language different than Conjecture 1.1. For instance, following [**55**] and [**17**] one may write down the conjecture that the correlation functions of zeta zeroes are described by a sine-kernel determinant in the following way: suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfies (i) $f(x + (t,...,t)) = f(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ and (ii) f is compactly supported in the hyperplane $x_1 + \cdots + x_n = 0$. Then

$$\lim_{T\to\infty} \frac{2\pi}{T\log T} \sum_{\substack{0 \le \gamma_1,\ldots,\gamma_n \le T \\ \text{distinct}}} f\Big(\frac{\log T}{2\pi}\gamma_1, ..., \frac{\log T}{2\pi}\gamma_n\Big)$$
$$= \int_{\mathbb{R}^n} \delta\Big(\frac{x_1+\cdots+x_n}{n}\Big) f(x) \det_{n\times n}\Big(K(x_i - x_j)\Big)\, d^n x. \quad (92)$$

The condition that $f(x_1,...,x_k)$ is symmetric in $x_1,...,x_k$ is often added, but because both the left and right hand sides symmetrize the function this is not necessary. Likewise this conjecture is also often made not only for f which are compactly supported, but also for f which are rapidly decaying in the $x_1 + \cdots + x_n = 0$ hyperplane. This slightly more general claim can be inferred from the claim for f which are compactly supported in the hyperplane and we discuss this at the end of the appendix.

It is also possible to write the above in the form that for $g \in C_c(\mathbb{R}^{n-1})$,

$$\lim_{T\to\infty} \frac{2\pi}{T\log T} \sum_{\substack{0\le\gamma_1,...,\gamma_n\le T\\ \text{distinct}}} g\Big(\frac{\log T}{2\pi}(\gamma_2-\gamma_1),...,\frac{\log T}{2\pi}(\gamma_n-\gamma_1)\Big)$$
$$= \int_{\mathbb{R}^n} \delta(x_1) g(x_2,...,x_n) \det_{n\times n}\Big(K(x_i-x_j)\Big)\, d^n x. \quad (93)$$

It is easily seen that (92) and (93) are equivalent by noting that any such function $f(x_1,...,x_n)$ can be written $g(x_2-x_1,...,x_k-x_1)$ by letting $g(y_2,...,y_k) = f(0,y_2,...,y_k)$.

We now quickly outline how (92) and the version of the GUE Conjecture found in Conjecture 1.1 can be seen to be equivalent. It is a simple consequence of Theorem 9.1 that Conjecture 1.1 may be equivalently restated with the average from $[T,2T]$ replaced by an average from $[0,T]$. Then to pass from (92) to Conjecture 1.1, note that because η is compactly supported, the sum in (1) can be restricted to $0\le\gamma_1,...,\gamma_n\le T$ at the cost of a $o(1)$ error term, using routine estimates for the number of γ_i in an interval. But if the sum in (1) is restricted to $0\le\gamma_1,...,\gamma_n\le T$, it also follows in much the same way that the integral from $[0,T]$ can be extended to an integral from $(-\infty,\infty)$ again at the cost of a $o(1)$ error term. Now let

$$f(x_1,...,x_k) = \frac{\log T}{2\pi}\int_{-\infty}^{\infty} \eta(x_1 - \tfrac{\log T}{2\pi}t,...,x_k-\tfrac{\log T}{2\pi}t)\,dt, \quad (94)$$

and one sees that Conjecture 1.1 follows from (92).

In the other direction, to see that (92) follows from Conjecture 1.1 note that for compactly supported f satisfying $f(x+(t,...,t)) = f(x)$, if we define

$$\eta(x_1,...,x_n) = w\Big(\frac{x_1+\cdots+x_n}{n}\Big) f(x_1,...,x_n),$$

where $w\in C_c(\mathbb{R})$ is of mass 1, then the reader may check (94) still holds. The rest of the proof follows the same steps as before.

Finally we have promised to explain why (92) and (93) imply the same claims for functions which rapidly decay. To see this, one may use Proposition 9.4. Via that Proposition, Conjecture 1.1 can be extended to this wider class of test functions, and in the same way as above, one can see that Conjecture 1.1 is equivalent to (92) and (93) for this wider class.

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